1. Since $\sigma$ is increasing on the closed interval $[0, 2\pi]$, it is of bounded variation. (In fact $V_0(\sigma) = \sigma(2\pi) - \sigma(0)$.)

$\int_0^{2\pi} \frac{e^{it\tau} + \tau}{e^{it\tau} - \tau} d\sigma(t) + i\beta$

Or, put $\tau = 0$ in $f(\tau) = \int_0^{2\pi} \frac{e^{it\tau} + \tau}{e^{it\tau} - \tau} d\sigma(t) + i\beta$

to obtain $f(\tau) = \int_0^{2\pi} d\sigma(t) + i\beta$. Thus

$\int_0^{2\pi} d\sigma(t) = \text{Re} f(\tau)$, which proves that $V_0(\sigma) = \text{Re} f(\tau) < \infty$.

In the expression $f(\tau) = \int_{-\infty}^{\infty} \frac{1 + itx}{x + \tau} d\sigma(t) + \mu \tau + \lambda$,

with $\sigma$ increasing, put $\tau = i$:

$f(i) = \alpha + i\mu + \int_{-\infty}^{\infty} \frac{1 + itx}{x + i} d\sigma(t) = \alpha + i\mu + i\int_{-\infty}^{\infty} d\sigma(t)$.

Thus

$\int_{-\infty}^{\infty} d\sigma(t) = \text{Im} f(i) - \mu < \infty$,

which shows that $\sigma$ is of bounded variation.
2. (a) \[ \sigma(t) = \begin{cases} 0, & t \leq 0 \\ 1, & t > 0 \end{cases} \quad (\alpha = 0, \mu = \infty) \]

Then \[ \int_{-\infty}^{\infty} \frac{1 + t^2}{t - z} d\sigma(t) = \left. \frac{1 + t^2}{t - z} \right|_{t=-\infty}^{t=0} = -\frac{1}{z} \quad \checkmark \]

(b) \[ \sigma(t) = \begin{cases} 0, & t \leq -1 \\ 1, & -1 < t \leq 1 \\ 2, & 1 < t \end{cases} \quad (\alpha = 0, \mu = \infty) \]

Then \[ \int_{-\infty}^{\infty} \frac{1 + t^2}{t - z} d\sigma(t) = \left. \frac{1 + t^2}{t - z} \right|_{t=-1}^{t=1} + \left. \frac{1 + t^2}{t - z} \right|_{t=1}^{t=2} \]

\[ = \frac{1 - z}{-1 - z} + \frac{4z}{1 - z} = \frac{4z}{-z^2} \]

(c) \[ f(z) = \zeta \quad g(s) = -i \zeta \left( \frac{1 + s}{1 - s} \right) = 1 \]

By the representation theorem for analytic fuzzy in a nbhd of \( D \), we have

\[ 1 = \frac{1}{2\pi i} \int_{0}^{2\pi} e^{it + s} \text{Re}(g(s)) \, dt = \int_{0}^{2\pi} \frac{e^{it + s}}{e^{it} - s} \, d\theta(t) \]

where \( \theta(t) = \frac{t}{2\pi} \) on \([-\pi, \pi]\). With \( \sigma(t) = \beta(2\arccot(t)) = \frac{t}{\pi} \arccot(t) \), we obtain

\[ i = \int_{-\infty}^{\infty} \frac{1 + t^2}{t - z} d\sigma(t) \quad \checkmark \]
Let \( f(z) \) be analytic in \( \mathbb{C} \setminus B \) with \( \text{Re} f(z) \geq 0 \) and \( z \in \mathbb{C} \setminus B \). Define \( \Theta \) and \( D \),
\[ g(r) = f(\frac{1}{r}) , \]
which is analytic in \( D \setminus \Theta \) and has nonnegative real part, \( g \) has a Laurent expansion about \( r = 0 \):
\[ g(r) = \sum_{n=-N}^{\infty} a_n r^n \]

If \( N = -\infty \), then \( g \) has an essential singularity at 0 and thus takes on all values in \( \mathbb{C} \setminus \Theta \) for some \( \Theta \in \mathbb{C} \) in each neighborhood of 0, contradicting \( \text{Re} g \geq 0 \).

Thus \( N \neq -\infty \), so \( g \) has a pole of order \( N \geq 0 \) (assume \( a_N \neq 0 \))
\[ g(r) = \frac{1}{r^N} h(r) \]

with \( h(0) \neq 0 \). Since \( h \) is continuous at 0 and \( h(0) \neq 0 \),
\[ \text{Re} \left( \frac{1}{r^N} h(r) \right) \]
takes on both positive and negative values in each punctured neighborhood of 0, contradicting \( \text{Re} g \geq 0 \).

Thus \( N = 0 \) and \( g \) has a removable singularity at 0.

Theorem 1 now applies:

Could include more details.
\( g(z) = e^{i\beta} + \int_{0}^{2\pi} \frac{e^{iz} + e^{-iz}}{e^{iz} - e^{-iz}} d\sigma(t) \) for some increasing function \( \sigma \) on \([0, 2\pi]\).

So \( f(z) = g\left(\frac{1}{z}\right) = e^{i\beta} + \int_{0}^{2\pi} \frac{e^{i\frac{\sigma(t)}{z}} + e^{-i\frac{\sigma(t)}{z}}}{z - e^{-i\frac{\sigma(t)}{z}}} d\sigma(t) \)

\[ = e^{i\beta} + \int_{0}^{2\pi} \frac{z + e^{-i\frac{\sigma(t)}{z}}}{z - e^{-i\frac{\sigma(t)}{z}}} d\sigma(t) \]

which is the representation we seek.

Now, assuming this representation of some function \( f \) defined in \( \mathbb{C}(\mathbb{D}) \), the transformation \( z \mapsto \frac{1}{z} \)
converges it into the form (7) for \( g(z) = f\left(\frac{1}{z}\right) \),
which is, by Theorem, analytic with \( \Re g(z) \geq 0 \).

Thus \( f(z) = g\left(\frac{1}{z}\right) \) is analytic with \( \Re f(z) \geq 0 \).