LECTURES ON SCATTERING RESONANCES

VERSION 0.01

MACIEJ ZWORSKI

DEPARTMENT OF MATHEMATICS
UC BERKELEY

Preface

These notes are an attempt to organize lectures (as they progress) given at Université de Paris-Nord in the Spring of 2011. The author is grateful for the support through his *Chair d'Excellence* at the Laboratoire Analyse, Géométrie et Applications there.

Contents

1. Introduction

1.1. RESONANCES IN SCATTERING THEORY.

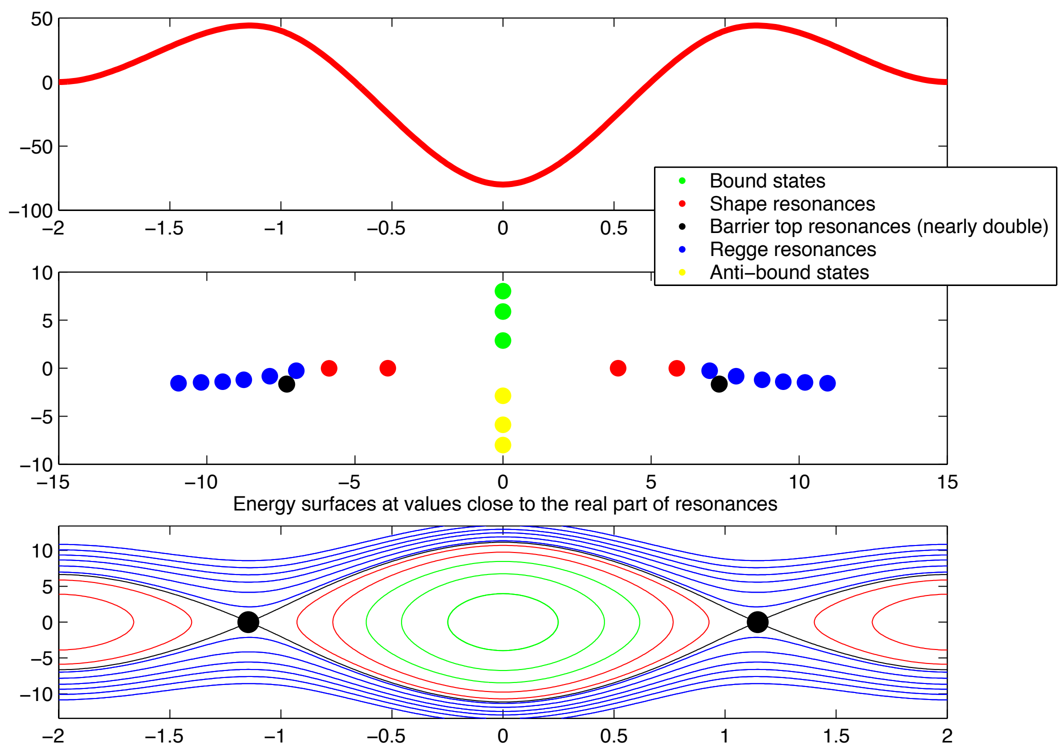

Figure 1. Resonances corresponding to different dynamical phenomena

Scattering resonances are the replacement of discrete spectral data for problems on non-compact domains.

1.2. SEMICLASSICAL STUDY OF RESONANCES.

Figure 1 shows some of the principles in dimension one.

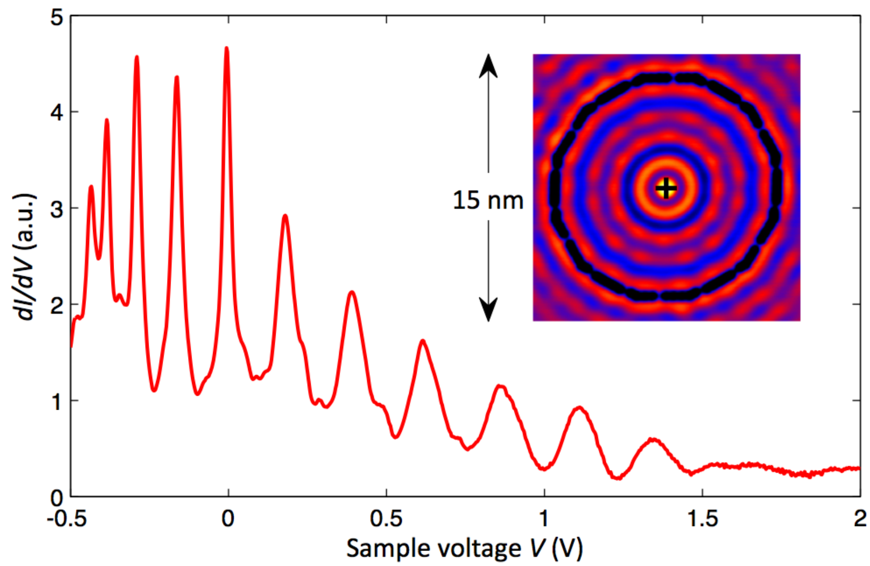

FIGURE 2. An STM spectrum is a plot of dI/dV (I being the current) as a function of bias voltage V. According to the basic theory of STM, this reflects the sample density of states as a function of energy with respect to the Fermi energy (at $V = 0$). This spectrum shows the series of surface state electron resonances in the center of a circular quantum corral on Cu(111). The bulk bands contribute to a gradually varying background in this spectrum. The setpoint was $V_0 = 1$V and $I_0 = 10$nA and the modulation voltage was $V_{\rm rms} = 4$mV. Inset: a low-bias topograph of the corral studied ($17 \times 17\text{nm}^2$, $V = 10$mV, $I = 1$nA). The corral is made from 84 CO molecules adsorbed to Cu(111) and has an average radius of 69.28 Å. The large amplitude in the center of the topograph is a re ection of the sharp peak seen in the spectrum at $V = 0$.

1.3. SOME EXAMPLES.

Although these lecture notes are intended for mathematical audience and concentrate on rigorous presentation, a physical motivation plays an essential rôle in the study of scattering resonances. We present here a few recent examples.

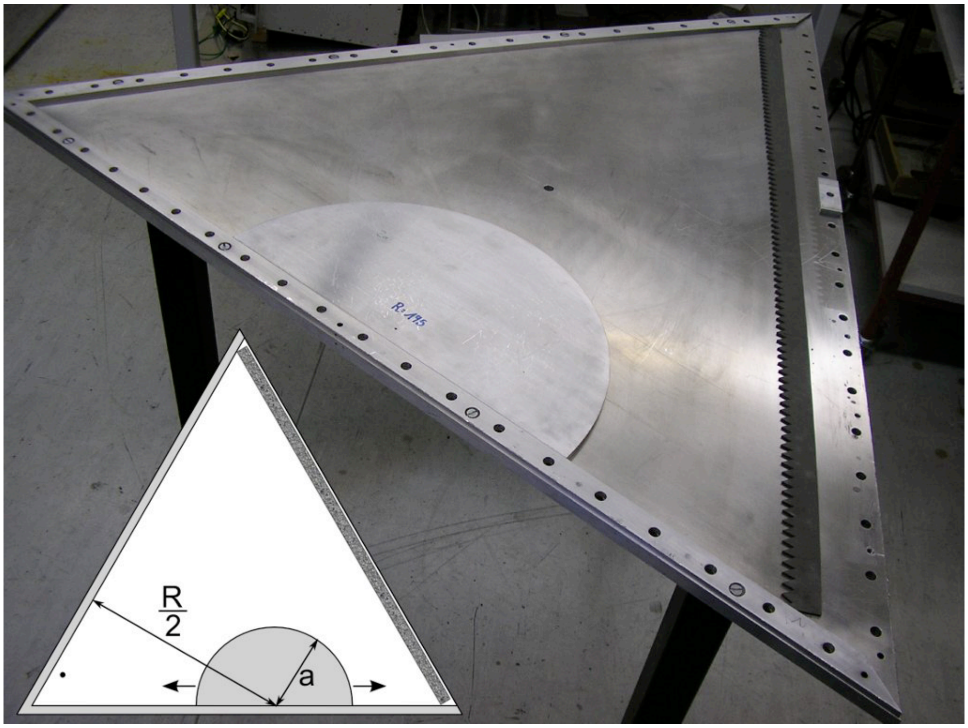

FIGURE 3. The experimental set-up of the Marburg quantum chaos group

`http://www.physik.uni-marburg.de`

for the five disc, symmetry reduced, system. The hard walls correspond to the Dirichlet boundary condition, that is to odd solutions (by reflection) of the full problem. The absorbing barrier, which produces negligible reflection at the considered range of frequencies, models escape to infinity.

Figure 2 shows resonance peaks for a scanning tunneling microscope experiment where a circular quantum corral of CO molecules is constructed – see [1] and references given there. Figure 3 shows an experimental set-up for studying resonances in microwave scattering.

Figure 4 shows a MEMS resonator. The numerical calculations in that case are based on the complex scaling technique adapted to the finite element methods, and known as the method of *perfectly matched layers*.

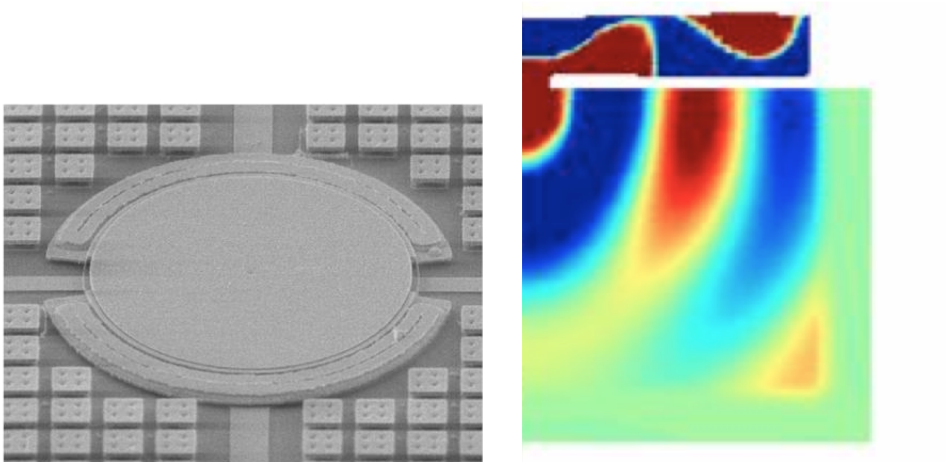

FIGURE 4. A MEMS device on top has resonances investigated using the complex scaling/perfectly matched layer methods [Bi-Go]. A numerically constructed resonant mode is shown on the right.

1.4. **OVERVIEW.** Chapter 1 covers basic theory of resonances in dimension one. The basic concepts such as the definition of outgoing solutions, meromorphic continuation of the resolvent, the relation of resonances to the scattering matrix, trace formulæ, and resonant expansions of waves.

2. **Scattering resonances in dimension one**

In the simplest and almost explicit setting of one dimensional scattering we can already see many of the features of the theory. In particular, various notions can be explained in a very intuitive setting. Technically, there are also many advantages: we are dealing with ordinary differential equations, methods of complex analysis apply particularly well, trace class properties hold nicely.

We consider the following class of operators:

$$P_V = D_x^2 + V(x) \,, \quad D_x := \frac{1}{i}\partial_x \,, \quad V \in L^\infty_{\rm comp}(\mathbb{R}) \,.$$

The stationary Schrödinger equation then is

$$(P_V - z)u = f \,, \quad z \in \mathbb{C} \,, \quad f \in L^\infty_{\rm comp}(\mathbb{R}) \,, \tag{2.1}$$

while the dynamical equation is given by

$$(i\partial_t - P_V)v = F \,, \quad v|_{t=0} = v_0 \,. \tag{2.2}$$

A solution to the stationary equation (2.1) produces solution to (2.2) corresponding to the evolution of the state u:

$$v(t,x) := e^{-izt}u(x) \,, \quad v_0(x) = u(x) \,, \quad F(x,t) = -e^{-izt}f(x) \,. \tag{2.3}$$

Outside the support of V and f, say for $|x| \geq R$, the solutions of (2.2) are given by

$$u(x) = A_\pm e^{i\sqrt{z}t} + B_\pm e^{-i\sqrt{z}t} \,, \quad \pm x \geq R \,.$$

To consider the dependence on z we have to choose a branch of $\sqrt{z}$. We consider $\sqrt{z}$ defined on $\mathbb{C} \setminus [0,\infty)$ with

$$\pm \lim_{\epsilon \to 0+} \sqrt{z \pm i\epsilon} =: \pm\sqrt{z \pm i0} > 0 \,, \quad z \in (0,\infty) \,.$$

When considering $z \in (0,\infty)$ we write $\sqrt{z} = \sqrt{z + i0}$.

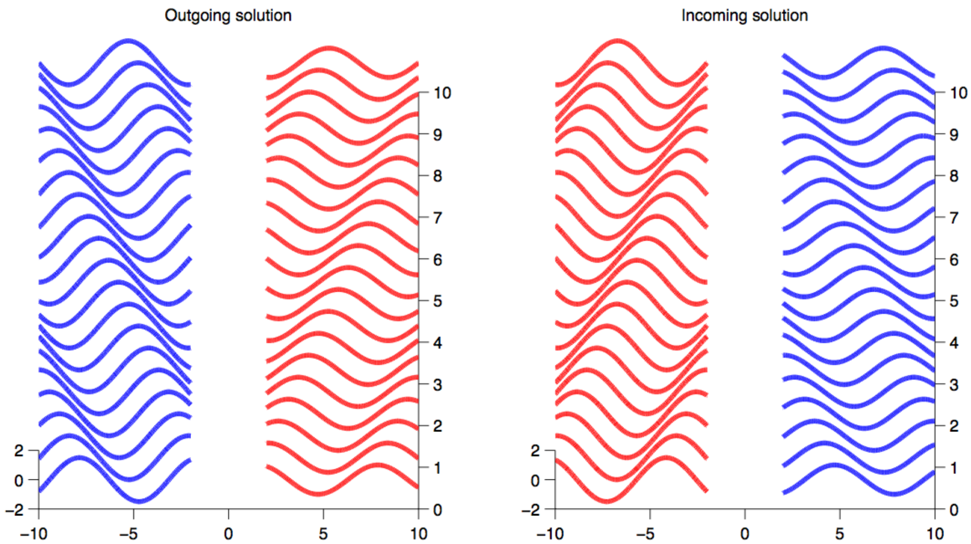

FIGURE 5. Schematic representation of the outgoing (left) and incoming (right) solutions to (2.3).

Outgoing and incoming solutions. A solution to (2.1) with $z > 0$ is called *outgoing* if

$$u(x) = B_- e^{-i\sqrt{z}x}\,, x < -R\,, \quad u(x) = A_+ e^{i\sqrt{z}x}\,, x > R\,. \tag{2.4}$$

This corresponds to v given by (2.3) moving away from the support of $V(x)$ – see Figure 5. We also note that using our convention

$$\operatorname{Im} z > 0 \implies u(x) \in L^2(\mathbb{R})\,.$$

Similarly, the solution to (2.1) is callled *incoming* if

$$u(x) = A_- e^{i\sqrt{z}x}\,, x < -R\,, \quad u(x) = B_+ e^{-i\sqrt{z}x}\,, x > R\,.$$

Although the physical motivation illustrated in Figure 5 disappears when $z \notin (0,\infty)$ we will still use the notions of outgoing and incoming solutions as defined above, paying attention to our $\sqrt{z}$ convention,

In Section 2.1 we will address the problem of constructing outgoing (or incoming solutions) to (2.1). That will need to a natural definition of resonances.

So far we have provided motivation in terms of the Schrödinger equation. We can also consider the wave equation:

$$(-\partial_t^2 - P_V)v = F\,, \quad v|_{t=0} = v_0\,, \quad \partial_t v|_{t=0} = v_1\,. \tag{2.5}$$

In that case the stationary equation, formally obtained by taking the Fourier transform in t, is given by

$$(P_V - \lambda^2)u = 0\,, \quad \lambda \in \mathbb{C}\,. \tag{2.6}$$

In this case the convention regarding the sign of λ in the definition of outgoing and incoming solutions is arbitrary. We choose a convention consistent with the choice of $\sqrt{z}$ above:

$$\lambda^2 = z\,, \quad \lambda = \sqrt{z}\,.$$

In particular,

$$\lambda > 0 \implies \sqrt{(\pm\lambda + i0+)^2} = \pm\lambda\,,$$

and the outgoing solution to (2.6) is supposed to satisfy

$$u(x) = B_- e^{-i\lambda x}\,, x < -R\,, \quad u(x) = A_+ e^{i\lambda x}\,, x > R\,. \tag{2.7}$$

We now have

$$\mathrm{Im}\lambda > 0 \implies u(x) \in L^2(\mathbb{R})\,.$$

The solutions to (2.6) with $f = 0$ are the eigenfunctions of P_V corresponding to eigenvalues λ^2.

We will use the λ convention in this chapter. In Section 2.2 the connection with the wave equation will be made precise and rigorous.

2.1. **Meromorphic continuation.**

In this section we will consider solving (2.6) for $\lambda \in \mathbb{C}$, with u outgoing, that is satisfying (2.7).

First we consider the case of $V = 0$. In that case $u(x)$ is given by an explicit formula:

$$u(x) = \frac{i}{2\lambda} \int_{\mathbb{R}} e^{i\lambda|x-y|} f(y) dy\,.$$

For $\mathrm{Im}\lambda > 0$ this gives the integral kernel of the *free resolvent*:

$$(D_x^2 - \lambda^2)^{-1}(x,y) = \frac{i}{2\lambda} e^{i\lambda|x-y|}\,,$$
$$(D_x^2 - \lambda^2)^{-1} \;:\; L^2(\mathbb{R}) \longrightarrow L^2(\mathbb{R})\,, \quad \mathrm{Im}\lambda > 0\,.$$

We should stress that for $\mathrm{Im}\lambda < 0$

$$(D_x^2 - \lambda^2)^{-1}(x,y) = -\frac{i}{2\lambda} e^{-i\lambda|x-y|}\,,$$
$$(D_x^2 - \lambda^2)^{-1} \;:\; L^2(\mathbb{R}) \longrightarrow L^2(\mathbb{R})\,, \quad \mathrm{Im}\lambda < 0\,.$$

We write

$$R_0(\lambda) := (D_x^2 - \lambda^2)^{-1}\,, \quad \mathrm{Im}\lambda > 0\,,$$
$$R_0(\lambda)(x,y) = \frac{i}{2\lambda} e^{i\lambda|x-y|}\,. \tag{2.8}$$

From this expression we see that $R_0(\lambda)(x,y)$ is a meromorphic function of $\lambda \in \mathbb{C}$ defining an operator $C_{\mathrm{c}}^\infty(\mathbb{R}) \to C^\infty(\mathbb{R})$ which is not bounded on L^2 for $\mathrm{Im}\lambda \leq 0$.

We summarize these in

THEOREM 2.1 (Meromorphic continuation of the resolvent 0). *The operator $R_0(\lambda)$ defined by (2.8) for $\mathrm{Im}\lambda > 0$ extends to a meromorphic family of operators for $\lambda \in \mathbb{C}$:*

$$R_0 := L^2_{\mathrm{comp}} \longrightarrow L^2_{\mathrm{loc}}\,.$$

We have

$$\|R_0(\lambda)\|_{L^2\to L^2} = \frac{1}{d(\lambda^2, \mathbb{R}_+)} \leq \frac{1}{|\lambda|\mathrm{Im}\lambda}\,, \quad \mathrm{Im}\lambda > 0\,,$$

and for $\rho \in C_{\mathrm{c}}^\infty(\mathbb{R})$, $\mathrm{supp}\rho \subset [-L, L]$

$$\|\rho R_0(\lambda)\rho\|_{L^2(\mathbb{R}^n)\to H^j(\mathbb{R}^n)} \leq C_L e^{2L(\mathrm{Im}\lambda)_-}|\lambda|^{j-1}\,, \tag{2.9}$$

where $x_- = 0$ for $x \geq 0$ and $x_- = -x$ for $x < 0$.

For $V \neq 0$ we have a result which shows that the resolvent of $P_V = D_x^2 + V(x)$ also has a meromorphic continuation.

THEOREM 2.2 (Meromorphic continuation of the resolvent I). *Suppose that $V \in L^\infty_{\mathrm{comp}}(\mathbb{R}; \mathbb{C})$. Then the*

$$R_V := (D_x^2 + V - \lambda^2)^{-1} \;:\; L^2 \longrightarrow L^2\,, \quad \mathrm{Im}\lambda > 0\,,$$

is a meromorphic family of operators with singularities contained in $D(0, R_V)$ for some V.

It extends to a meromorphic family of operators for $\lambda \in \mathbb{C}$:

$$R_V := L^2_{\mathrm{comp}} \longrightarrow L^2_{\mathrm{loc}}\,.$$

If λ_0 is a singularity of $\lambda \mapsto R_V(\lambda)$ then there exists a unique (up a multiplicative constant) outgoing solution u to $(P_V - \lambda_0^2)u = 0$.

Proof. 1. We first construct $R_V(\lambda)$ for $\mathrm{Im}\lambda \gg 1$. If the inverse $R_V(\lambda) = (P_V - \lambda^2)^{-1}$ existed then

$$R_V(\lambda) - R_0(\lambda) = -R_V(\lambda) V R_0(\lambda)$$

and hence

$$R_V(\lambda)(I + V R_0(\lambda)) = R_0(\lambda) .$$

For $\mathrm{Im}\lambda \gg 1$, $\|V R_0(\lambda)\|_{L^2 \to L^2} \leq \|V\|_\infty (\mathrm{Im}\lambda)^{-2} \leq 1/2$, and hence $I + V R_0(\lambda 0$ is invertible. That means that

$$R_V(\lambda) := R_0(\lambda)(I + V R_0(\lambda))^{-1} \tag{2.10}$$

is the desired inverse. For $\mathrm{Im}\lambda > 0$ the operator $V R_0(\lambda)$ is compact and hence we can apply Theorem C.4 to see that $R_V(\lambda) \ : \ L^2(\mathbb{R}) \to L^2(\mathbb{R})$ is a meromorphic family of operators in $\mathrm{Im}\lambda > 0$.

2. To obtain continuation of $\mathbb{C}$, choose $\rho \in C^\infty_{\mathrm{c}}$ which is equal to 1 on $\mathrm{supp}V$. In particular, $\rho V = V$. We define the following meromorphic family of operators in $\mathbb{C}$:

$$K(\lambda) := V R_0(\lambda) . \tag{2.11}$$

The only pole of $K(\lambda)$ is at $\lambda = 0$. Since $(1 - \rho) K(\lambda) = 0$ we have

$$\begin{aligned} (I + K(\lambda)(1-\rho))^{-1} &= (I - K(\lambda)(1-\rho)) \\ (I + K(\lambda))^{-1} &= ((I + K(\lambda)(1-\rho))(I + K(\lambda)\rho))^{-1} \\ &= (I + K(\lambda)\rho)^{-1}(I - K(\lambda)(1-\rho)) , \end{aligned}$$

where the second identity holds when $I + K(\lambda)\rho$ is invertible which we know already for $\mathrm{Im}\lambda \gg 0$ by a Neumann series argument.

3. We conclude that for $\mathrm{Im}\lambda > 0$

$$R_V(\lambda) = R_0(\lambda)(I + V R_0(\lambda)\rho)^{-1}(I - V R_0(\lambda)(1-\rho)) . \tag{2.12}$$

The operator $V R_0(\lambda)\rho$ is compact on L^2 since

$$V R_0(\lambda)\rho = V(\rho R_0(\lambda)\rho) \ \text{ and } \ \rho R_0(\lambda)\rho : L^2 \to H^2(\mathrm{supp}\rho) .$$

Hence, Therem C.4 gives the global meromorphic continuation of $(I + V R_0(\lambda)\rho)^{-1}$. We also observe that

$$I + V R_0(\lambda)(1-\rho) \ : \ L^2_{\mathrm{comp}}(\mathbb{R}) \to L^2_{\mathrm{comp}}(\mathbb{R}) ,$$

and

$$(I + V R_0(\lambda)\rho)^{-1} \ : \ L^2_{\mathrm{comp}}(\mathbb{R}) \to L^2_{\mathrm{comp}}(\mathbb{R}) .$$

The last property can be checked for $\mathrm{Im}\lambda \gg 1$ using the Neumann series argument: if $\chi\rho = \rho$, $\tilde\chi\chi = \chi$ then

$$(1 - \tilde\chi)(I + V R_0(\lambda)\rho)^{-1}\chi = 0 , \quad \mathrm{Im}\lambda \gg 0 ,$$

and this remains true for all λ by analytic continuation.

Combining these facts with the expression for R_V given in (2.12) we obtain the meromorphy of $R_V(\lambda)$ for $\lambda \in \mathbb{C}$ as a family of operators $L^2_{\rm com} \to L^2_{\rm loc}$.

4. We finally prove that a pole of $R_V(\lambda)$ at λ_0 gives an outgoing solution to $(P - \lambda_0^2)u = 0$. From (2.12) we see that having a pole of $R_V(\lambda)$ implies that $I + VR_0(\lambda)\rho$ is not invertible. Since $VR_0(\lambda)\rho$ is compact, $I + VR_0(\lambda)\rho$ is a Fredholm operator of index zero (see Section C.2). That means that there exists $v \in L^2$ such that $v = -VR_0(\lambda_0)\rho v$. Since $\rho V = V$ we see that $\rho v = v$ and hence $v \in L^2_{\rm com}$ and $v = -VR_0(\lambda_0)v$. Putting $u := R_0(\lambda_0)v$ we see that u is outgoing and that

$$(P - \lambda_0^2)u = (D^2 - \lambda_0^2)R_0(\lambda_0)v + VR_0(\lambda)v = v + VR_0 v = 0\,.$$

Since we are dealing with an ordinary differential equation the solution equal to $ae^{i\lambda_0 x}$ for $x > R$ is unique up to a multiplicative constant. $\square$

DEFINITION. We call the poles of $R_V(\lambda)$ **scattering resonances** or simply **resonances.** The multiplicity of a resonance at λ is defined as follows:

$$m_R(\lambda) := \operatorname{rank} \oint_\lambda R(\zeta)d\zeta\,, \tag{2.13}$$

where the integral is over a small circle containing no other poles of R_V.

When λ is not a resonance we put $m_R(\lambda) = 0$ which is of course consistent with the above definition. In Section 2.6 we will investigate the structure of the singular part of $R_V(\lambda)$ in more detail.

REMARKS.

1. When $V \in L^\infty_{\rm comp}(\mathbb{R}, \mathbb{R})$ then the operator P_V is self-adjoint and the existence of $R_V(\lambda)$, $\mathrm{Im}\lambda > 0$, as a meromorphic operator on L^2 follows from the spectral theorem. The poles occur at $i\sqrt{-E_j}$ where E_j are the negative eigenvalues of P_V – see Figure 1.

2. We also have the following basic fact valid for real valued potentials:

$$V \in L^\infty_{\rm comp}(\mathbb{R}; \mathbb{R}) \implies m_R(\lambda) = 0\,,\ \lambda \in \mathbb{R} \setminus \{0\}\,. \tag{2.14}$$

Proof of 2.14: We need to show that there are no outgoing solutions to $(P_V - \lambda^2)u = 0$ for λ real and non-zero (at $\lambda = 0$ the example of

$V = 0$ shows that a pole is possible and the outgoing solution is given by $u = 1$). Since V is real $\bar{u}$ is also a solution. Using the notation of (2.7) we calculate the Wronskians:

$$W(u, \bar{u}) := \begin{vmatrix} u & \bar{u} \\ u_x & \bar{u}_x \end{vmatrix} = \begin{cases} 2i\lambda |B_-|^2 \,, & x < -R \,, \\ -2i\lambda |A_+|^2 \,, & x > R \,. \end{cases}$$

But this is impossible for $\lambda \neq 0$ and $u \not\equiv 0$. $\square$

3. Reality of V or, equivalently, self-adjointness of P_V imply the following symmetry of resonances:

$$V \in L^\infty_{\rm comp}(\mathbb{R}; \mathbb{R}) \implies m_R(\lambda) = m_R(-\bar{\lambda}) \,, \ \lambda \in \mathbb{R} \setminus \{0\} \,. \tag{2.15}$$

In fact, we will check that $R_V(-\bar{\lambda})^* = R_V(\lambda)$. Since both sides are meromorphic in λ we only need to check that $R_V(-\lambda)^* = R_V(\lambda)$ for $\lambda \in \mathbb{R}$. Using the correspondence between λ and z that follows from $((P_V - z - i0)^{-1})^* = (P_V - z + i0)^{-1}$.

The next result makes the structure of the singular part of $R_V(\lambda)$ more precise.

THEOREM 2.3 (Singular part of $R_V(\lambda)$ I).
1) Suppose $m_R(\mu) > 0$, $\mu \neq 0$. Then, there exist linearly independent $u_j \in H^2_{\rm loc}(\mathbb{R})$, $j = 1, \cdots, m_R(\mu)$, such that u_1 is outgoing and

$$(P_V - \mu) u_1 = 0 \,, \quad (P_V - \mu^2) u_j = u_{j-1} \,, \tag{2.16}$$

$1 < j \leq m_R(\mu)$.

We also have

$$R_V(\lambda) = \sum_{k=1}^{m_R(\mu)} \frac{(P - \mu^2)^{k-1}}{(\lambda^2 - \mu^2)^k} \Pi_\mu + A(\lambda, \mu) \,, \tag{2.17}$$

where $\lambda \mapsto A(\lambda, \mu)$ is holomorphic near μ,

$$\Pi_\mu = \frac{1}{2\pi i} \oint_\mu R_V(\lambda) 2\lambda d\lambda \,,$$

and

$$(P_V - \mu^2)^{m_R(\mu)} \Pi_\mu = 0 \,, \quad {\rm Im} \Pi_\mu = {\rm span} \{u_1, \cdots, u_{m_R(\mu)}\} \,. \tag{2.18}$$

2) Suppose that $V \in L^\infty_{\rm comp}(\mathbb{R}; \mathbb{R})$ and that $m_R(0) > 0$. Then $m_R(0) = 1$ and

$$R_V(\lambda) = \frac{\Pi_0}{\lambda} + A(\lambda) \,,$$

where $\lambda \mapsto A(\lambda)$ *is holomorphic near* 0*, and*

$$\Pi_0 = u \otimes u\,, \quad u = c_\pm \neq 0\,, \; \pm x \gg 0\,, \quad P_V u = 0\,. \tag{2.19}$$

REMARKS. 1. The reason for restricting out attention at $\mu = 0$ to real V's, that is to selfadjoint operators P_V, lies in the fact that we need the specifics about the resonance zero only for resonance expansions and trace formulæ. In both cases we assume selfadjointness of P_V so that we can use the spectral theorem.

2. In Section 2.6 we will find an interpretation of Π_μ, $\mu \neq 0$, as a projection.

Proof. 1. From the general result about meromorphic continuation we know that, for some K and finite rank operators A_k,

$$R_V(\lambda) = \sum_{k=1}^{K} \frac{A_k}{(\lambda^2 - \mu^2)^k} + A(\lambda, \mu)\,, \quad \mu \neq 0\,,$$

where $m_R(\mu) = \operatorname{rank} A_1$ and

$$A_1 = \Pi_\mu := \frac{1}{2\pi i} \oint_\mu R_V(\lambda) 2\lambda d\lambda\,.$$

2. We now consider the equation $(P_V - \lambda^2) R_V(\lambda) = 0$ near $\lambda = \nu$: modulo terms holomorphic near μ we have

$$\begin{aligned}(P_V - \lambda^2) R_V(\lambda) &\equiv \sum_{k=1}^{K} \left(\frac{(P_V - \mu^2) A_k}{(\lambda^2 - \mu^2)^k} - \frac{A_k}{(\lambda^2 - \mu^2)^{k-1}} \right) \\ &\equiv \sum_{k=1}^{K} \frac{(P_V - \mu^2) A_k - A_{k+1}}{(\lambda^2 - \mu^2)^k}\,,\end{aligned}$$

where we use the convention that $A_k = 0$ for $k > K$.
It follows that $A_{k+1} = (P - \mu^2) A_k$ which shows that (2.17) holds.

3. We now need to show the existence of u_j's satisfying (2.16) and (2.18). That includes the statement that $K = m_\mu(R)$.

The operator $(P_V - \mu^2)$ commutes with Π_μ and $(P_V - \mu^2)^K \Pi_\mu = 0$. Hence

$$P_V - \mu^2 : \operatorname{Im}\Pi_\mu \to \operatorname{Im}\Pi_\mu\,,$$

is nilpotent and we can put it into a Jordan normal form. That means that there exists a basis of $\mathrm{Im}\Pi_\mu \subset H^2_{\mathrm{com}}(\mathbb{R})$ such that

$$u_{\ell,j}\,, \quad 1 \le \ell \le L\,, \quad 1 \le j \le k_\ell\,, \quad \sum_{\ell=1}^{L} k_\ell = K\,,$$
$$(P_V - \mu^2)u_{\ell,j} = u_{\ell,j-1}\,, \quad 1 \le j \le k_\ell\,, \quad u_{\ell,0} := 0\,.$$

Arguing as at the end of the proof of Theorem 2.2 we see that $u_{\ell,1}$ is outgoing. But then it is unique up to a multiplicative constant. This shows that $L = 1$ and that $u_j := u_{1,j}$ satisfy (2.16). Since the dimension of $\mathrm{Im}\Pi_\mu$ is equal to $m_R(\mu)$ be definition we obtain $K = m_R(\mu)$ and (2.18).

4. For the study of $\mu = 0$ we assumed that P_V is selfadjoint. Hence for $\mathrm{Im}\lambda > 0$, $|\lambda| \ll 1$ (so that we avoid possible eigenvalues),

$$\|R_V(\lambda)\| = \frac{1}{d(\lambda^2, \mathbb{R}_+)} \le \frac{1}{|\lambda|\mathrm{Im}\lambda}\,,$$

which shows that $m_R(0) \le 2$, and

$$R_V(\lambda) = \frac{A_2}{\lambda^2} + \frac{A_1}{\lambda} + A(\lambda)\,.$$

Applying $(P_V - \lambda^2)$ to both sides and letting $\lambda \to 0$ we see that

$$P_V A_j = A_j P_V = 0\,, \quad P_V A(0) = I + A_2\,.$$

This means that $A_1 = a_1 u \otimes u$, $A_2 = a_2 u \otimes u$, where u satisfies (2.19).

Also, for $\psi \in C^\infty_{\mathrm{c}}(\mathbb{R})$,

$$\|(A_2 + \lambda A_1 + \lambda^2 A(\lambda))\psi\|_{L^2} \le \frac{|\lambda|^2}{d(\lambda^2, \mathbb{R}_+)}\|\psi\|_{L^2}\,, \quad \mathrm{Im}\lambda > 0\,.$$

Hence, letting $\lambda = it$, $t \to 0+$,

$$\|A_2\psi\|_{L^2} \le \|\psi\|_{L^2}\,.$$

Since $a_2 u \otimes u$ for u satisfying (2.19) is not bounded on L^2 if $a_2 \neq 0$ we see that $A_2 = 0$. We can now choose u so that $a_1 = 1$ and this proves part 2) of the theorem. $\square$

EXAMPLE. We present a natural family of potentials which have resonances of multiplicity 2 for some values in the family. This is illustrated in Figure 6.

Consider a potential $V \in C^1\mathrm{c}(\mathbb{R};\mathbb{R})$, $\mathrm{supp}V \subset [-a, a]$ with the property that $V(x) < -c < 0$ for, say, $x \in (-b, b)$, $0 < b < a$. We then

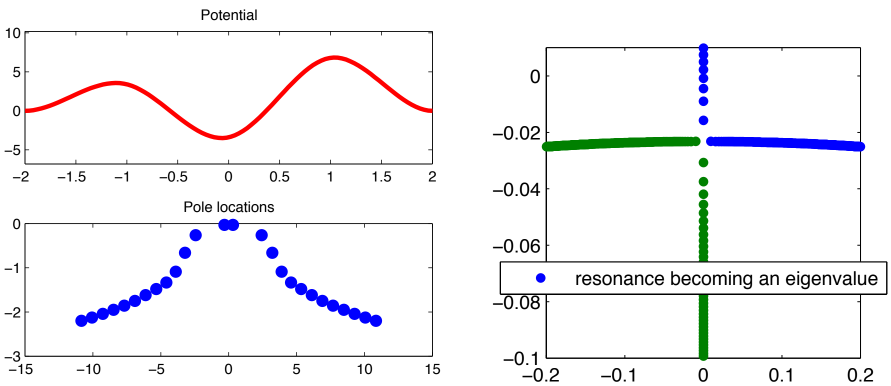

FIGURE 6. We consider resonances for τV where V, and its resonances are shown on the left. Taking $1 < \tau < 1.12$, we see two continuous families of resonances meeting on $i\mathbb{R}_-$. Pseudospectral effects due to the non-normal nature of R_V at the point of multiplicity two (see Theorem 2.3) make the motion very rapid near at the bifurcation. Hence the double resonance is hard to pinpoint numerically. The specific potential and it resonances was obtained using

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splinepot(3.4*[0,1,-1,2,0],[-2,-1,0,1,2])
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see [Bi-Zw].

consider a family of potentials τV, $\tau \geq 1$, that is we vary the coupling constant in the Schrödinger operator

$$P_\tau := D_x^2 + \tau V(x) .$$

By applying min-max methods directly (see Theorem B.6) or by using semiclassical Weyl law (with $h^2 = 1/\sqrt{\tau}$ – see for instance [D-S, Theorem 9.6]) we see that the number of negative eigevalues of P_τ grows (proportionally to $\sqrt{\tau}$) as τ increases.

The construction of $R_{\tau V}(\lambda)$ also shows that for any R, resonances in $D(0,R)$ are continues as functions of τ – see Chapter 6 for detailed arguments. This means that eigenvalues, that is resonances on $i\mathbb{R}_+$, are obtained, as τ increases from a continuous family of resonances passing through zero.

In view of the symmetry of resonances with respect to the real axis given in (2.15), and of the simplicity of the resonance at $\lambda = 0$ given in

Theorem 2.3, it means that two resonances meet on $-i\mathbb{R}_+$ before splitting, and one of them moving through 0 to become an eigenvalue. This provides a simple example of resonances, $\mu \in i\mathbb{R}_-$ for which $m_R(\mu) = 2$.

The multiplicity of a resonance can also be described using the following determinant:

$$D(\lambda) := \det(I + VR_0(\lambda)\rho)\,. \tag{2.20}$$

where $\rho \in L^\infty_{\rm comp}$ and $\rho V = V$.

We note that $D(\lambda)$ is a meromorphic function of λ with a single pole at $\lambda = 0$. The multiplicity of a zero of $D(\lambda)$ is defined in the usual way and we have,

$$m_D(\lambda) := \frac{1}{2\pi i} \oint \frac{D'(\zeta)}{D(\zeta)} d\zeta\,, \tag{2.21}$$

where the integral is over a positively oriented circle which includes λ and no other pole or zero of $D(\lambda)$.

THEOREM 2.4 (Multiplicity of a resonance I). *The multiplicities defined by* (2.13) *and*(2.21) *are related as follows*

$$m_D(\lambda) = m_R(\lambda)\,, \quad \lambda \in \mathbb{C} \setminus \{0\}\,. \tag{2.22}$$

When $V \in L^\infty(\mathbb{R};\mathbb{R})$ *then*

$$m_D(0) = m_R(0) - 1\,. \tag{2.23}$$

Proof. The proof is based on the Gohberg-Sigal theory of residues for meromorphic families of operators, which is reviewed in Section C.4.

1. This time let us start with $\mu = 0$ (in which case we assume that V is real valued so that part 2) of Theorem 2.3 applies). We have

$$I + VR_0(\lambda)\rho = \frac{i}{2\lambda} V \otimes \rho + A(\lambda)\,,$$

where $A(\lambda)$ is holomorphic and compact. Part 2) of Theorem 1 shows that

$$(I + VR_0(\lambda)\rho)^{-1} = I - VR_V(\lambda)\rho = \frac{Vu \otimes \rho u}{\lambda^{m_R(0)}} + B(\lambda)\,,$$

where $B(\lambda)$ is holomorphic near 0, and $Vu \not\equiv 0$, $\rho u \not\equiv 0$, if $m_R(0) \neq 0$. In Theorem C.5 applied to $M(\lambda) = I + VR_0(\lambda)\rho$ and $M(\lambda)^{-1}$ we must have $N = 2$, $k_1 = m_R(0)$ and $k_2 = -1$ – see (C.12). Using (C.14) we obtain (2.23).

2. For $\mu \neq 0$, $m_R(\mu) \neq 0$, we use $(I + VR_0(\lambda)\rho)^{-1} = I - VR_V(\lambda)\rho$ and Theorem 2.3 to obtain

$$(I + VR_0(\lambda)\rho)^{-1} = \sum_{k=1}^{m_R(\mu)} \frac{V(P_V - \lambda^2)^{k-1}\Pi_\mu \rho}{(\lambda^2 - \mu^2)^k} + A(\lambda)\,,$$

where $A(\lambda)$ is holomorphic and compact near $\lambda = \mu$. The arguments based Section C.4 shows $m_R(\mu) = m_D(\mu)$ **More details needed here.** $\square$

2.2. **Expansions of scattered waves.**

A motivation for the study of resonances is the fact that they describe oscillations and decay of waves for problems on non-compact domains. In this sense they replace eigenvalues and Fourier series expansions. Except for Theorem 2.6 we assume in this section that V is real valued. That is because we want to use methods of spectral theory of self-adjoint operators.

To explain this consider $P_V = D_x^2 + V$ on $[a, b]$ with Dirichlet (or Neumann) boundary condition. Then the problem

$$\begin{cases} (P_V - \lambda^2)u = 0 & \text{on } (a, b) \\ u(a) = u(b) = 0 \end{cases}$$

has a set of distinct solutions

$$\begin{gathered} (i\sqrt{-E_k},\, v_k)\,, \quad (\lambda_j, u_j)\,, \\ E_N < \cdots < E_1 < 0 < \lambda_0^2 < \lambda_1^2 < \cdots \to \infty\,, \\ \int_a^b |u_j|^2 dx = \int_a^b |v_k|^2 dx = 1\,. \end{gathered}$$

We then consider the wave equation

$$\begin{cases} (D_t^2 - P_V)w = 0 & \text{on } \mathbb{R} \times (a, b) \\ w(0, x) = w_0(x)\,, \quad \partial_t w(0, x) = w_1(x) & \text{on } [a, b] \\ w(t, a) = w(t, b) = 0 & \text{on } \mathbb{R}\,. \end{cases}$$

It can be solved using the eigenfunction expansion (Fourier series in the case when $V \equiv 0$):

$$
\begin{aligned}
w(t,x) = & \sum_{k=1}^{N} \cosh(t\sqrt{-E_k}) a_k v_k(x) + \sum_{k=1}^{N} \sinh(t\sqrt{-E_k}) b_k v_k(x) \\
& + \sum_{j=0}^{\infty} \cos(t\lambda_j) c_j u_j(x) + \sum_{j=0}^{\infty} \sin(t\lambda_j) d_j u_j(x)
\end{aligned} \tag{2.24}
$$

where

$$
\begin{aligned}
a_k &= \int_a^b w_0(x)\bar{v}_k(x)dx \ , \qquad & b_k &= \int_a^b w_1(x)\bar{v}_k(x) \\
c_j &= \int_a^b w_0(x)\bar{u}_j(x)dx \ , \qquad & d_j &= \int_a^b w_1(x)\bar{u}_j(x)dx \ .
\end{aligned}
$$

We now give the analogue of (2.24) when $[a,b]$ is replaced by $\mathbb{R}$:

THEOREM 2.5 (Resonance expansions of scattering waves I). *Let $V \in L^\infty(\mathbb{R};\mathbb{R})$ and suppose that $w(t,x)$ is the solution of*

$$
\begin{cases}
(D_t^2 - P_V)w(t,x) = 0 \,, \\
w(0,x) = w_0(x) \in H^1_{\rm comp}(\mathbb{R}) \,, \\
\partial_t w(0,x) = w_1(x) \in L^2_{\rm comp}(\mathbb{R}) \,.
\end{cases} \tag{2.25}
$$

Let $E_N < \cdots < E_1 < 0$ be the negative eigenvalues of P_V and $\{\lambda_j\} \subset \{\mathrm{Im}\lambda < 0\}$ be the set of its resonances.

Then, for any $A > 0$,

$$
\begin{aligned}
w(t,x) = & \sum_{k=1}^{N} \cosh(t\sqrt{-E_k}) a_k v_k(x) + \sum_{k=1}^{N} \sinh(t\sqrt{-E_k}) b_k v_k(x) \\
& + \sum_{\mathrm{Im}\lambda_j > -A} \sum_{\ell=0}^{m_R(\lambda_j)-1} \lambda_j^\ell e^{-i\lambda_j t} w_{j,\ell}(x) + E_A(t) \,,
\end{aligned} \tag{2.26}
$$

where the second sum is finite,

$$
\sum_{\ell=0}^{m_R(\lambda_j)-1} \lambda_j^\ell e^{-i\lambda_j t} w_{j,\ell}(x) = \mathrm{Res}_{\lambda=\lambda_j} \left(\left(iR_V(\lambda)w_1 + \lambda R_V(\lambda)w_0 \right) e^{-i\lambda t} \right) \,, \tag{2.27}
$$

$$
(P_V - \lambda_j)^{k+1} w_{j,k} = 0 \,,
$$

and for any $K > 0$*, such that* $\operatorname{supp} w_j \subset [-K, K]$*, there exist constants* $C_{K,A}$ *and* $T_{K,A}$

$$\|E_A(t)\|_{H^2([-K,K])} \leq C_{K,A} e^{-tA} \left(\|w_0\|_{H^1} + \|w_1\|_{L^2} \right) , \quad t \geq T_{K,A} .$$

REMARK. We notice that the error term $E_A(t)$ is more regular for large times. That corresponds to propagation of singularites: when time is large all singularities leave a compact set. When $V \in C_{\mathrm{c}}^\infty(\mathbb{R})$ then an examination of the proof shows that we have the same bound with the right hand side replaced by $\|E_A(t)\|_{H^k([-K,K])}$ for any k.

Before proving Theorem 2.5 we need the existence of a resonance free region and an estimate for the resolvent:

THEOREM 2.6 (Resonance free regions I). *Suppose that*

$$V \in L^\infty_{\mathrm{comp}}(\mathbb{R}; \mathbb{C}) .$$

Then for any $\rho \in C_{\mathrm{c}}^\infty(\mathbb{R})$ *constants* A', C, T *depending on the support of* ρ *such that*

$$\|\rho R_V(\lambda) \rho\|_{L^2 \to H^j} \leq C |\lambda|^{j-1} \, e^{T |\operatorname{Im} \lambda|} , \quad j = 0, 1, 2 , \tag{2.28}$$

for

$$\operatorname{Im} \lambda \geq -A - \delta \log(1 + |\lambda|) , \quad |\lambda| > C_0 , \quad \delta > 1 / |\operatorname{chsupp} V| .$$

In particular there are only finitely many resonances in the region

$$\{ \lambda \, : \, \operatorname{Im} \lambda \geq -A - \delta \log(1 + |\lambda|) \} .$$

for any $A > 0$.

Proof. 1. First we recall estimate (2.9) for the free resolvent

$$\|\rho R_0(\lambda) \rho\|_{L^2 \to H^j} \leq C |\lambda|^{j-1} e^{T |\operatorname{Im} \lambda|} \tag{2.29}$$

for some constant T depending on the support of ρ, Since

$$\rho R_V(\lambda) \rho = \rho R_0(\lambda) \rho (I + V R_0(\lambda) \rho_1)^{-1} (1 - V R_0(\lambda)(1 - \rho_1) \rho)$$

where we assumed that $\rho = 1$ on $\operatorname{supp} V$, and $\rho_1 L^\infty_{\mathrm{com}}(\mathbb{R})$ is any function satisfying $\rho_1 V = V$, in particular

$$\rho_1 = 1\!\!1_I , \quad I = \operatorname{chsupp} V .$$

We see now that (2.28) holds in the region where, say,

$$\|V \rho_1 R_0(\lambda) \rho_1\|_{L^2 \to L^2} \leq \frac{1}{2} .$$

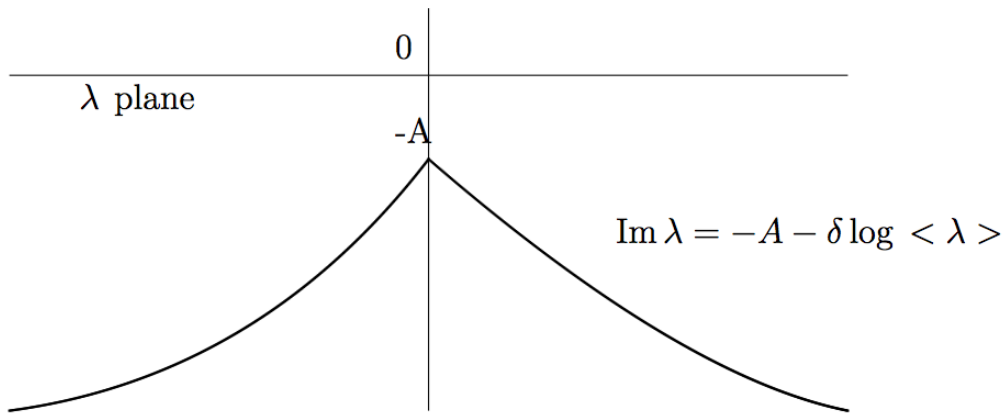

FIGURE 7. The contour used to obtain the resonance expansion.

For that we use (2.29) with $j = 0$ and obtain for $\mathrm{Im}\lambda > -A - \delta\log(1+|\lambda|)$,

$$\begin{aligned}\|V\rho_1 R_0(\lambda)\rho_1\|_{L^2\to L^2} &\leq C\|V\|_{L^\infty} e^{|I||\mathrm{Im}\lambda|}/|\lambda| \\ &\leq C\|V\|_\infty e^{A|I|+\delta|I|\log(1+|\lambda|)}/|\lambda| \\ &\leq C'\|V\|_\infty \||\lambda|^{-1+\delta|I|} \leq 1/2\,,\end{aligned}$$

a if $\delta < 1/|I|$ and $|\lambda| \geq R$. ☐

The idea for obtaining the expansion (2.26) is to deform the contour in the representation of the wave propagator based on the spectral theorem.

Proof of Theorem 2.5. 1. For simplicity, we assume that P_V has no negative eigenvalues as their contribution to (2.26) is clear. **For the moment we also assume that $\boldsymbol{m_R(0) = 0}$.**

Also, we will only consider (2.25) with $w_0 \equiv 0$ as the proof below works in the case $w_1 \equiv 0$ if we replace $\sin t\lambda/\lambda$ by $\cos t\lambda$ in the formula for $w(t,x)$.

2. With the above simplications understood, by the spectral theorem, the solution of (2.25) can be written as

$$w(t) = U(t)w_1 := \int_0^\infty \frac{\sin t\lambda}{\lambda}\, dE_\lambda(w_1)\,.$$

Using Stone's Formula to write dE_λ in terms of $R_V(\lambda)$, we get

$$dE_\lambda = \frac{1}{\pi i}(R_V(\lambda) - R_V(-\lambda))\lambda d\lambda \,,$$
$$P_V = \int_0^\infty \lambda^2 dE_\lambda \,, \quad I = \int_0^\infty dE_\lambda \,.$$

Hence

$$\begin{aligned} w(t) &= \frac{1}{\pi i}\int_0^\infty \sin t\lambda (R_V(\lambda) - R_V(-\lambda))w_1 d\lambda \\ &= \frac{1}{\pi i}\int_0^\infty \frac{e^{it\lambda} - e^{-it\lambda}}{2i}(R_V(\lambda) - R_V(-\lambda))w_1 d\lambda \qquad (2.30) \\ &= \frac{1}{2\pi}\int_{-\infty}^\infty e^{-it\lambda}(R_V(\lambda) - R_V(-\lambda))w_1 d\lambda \,, \end{aligned}$$

where we assumed that there is no resonance at $\lambda = 0$. To justify the convergence of the integral we assume that $w_1 \in H^2_{\rm comp}(\mathbb{R})$ as explained in the next step.

3. Now let $\rho \in C^\infty_{\rm c}(\mathbb{R})$ satisfy $\rho \equiv 1$ on the support of w_1. Choose R large enough so that all the resonances with $\mathrm{Im}\lambda > -A - \delta \log(1 + |\mathrm{Re}\lambda|)$ are contained in $|\lambda| \le R$. We deform the contour of integration in the above integral using the following curves:

$$\Gamma := \{\lambda - i(A + \epsilon\delta \log(1 + |\mathrm{Re}\lambda|)) \ : \ \lambda \in \mathbb{R}\} \,,$$
$$\Gamma_R := \Gamma \cap \{|\mathrm{Re}\lambda| \le R\} \,,$$
$$\gamma_R^\pm = \{\pm R - it \ : \ 0 \le t \le A + \epsilon + \delta \log(1 + R)\} \,, \quad \gamma_R := \gamma_R^+ \cup \gamma_R^- \,,$$
$$\gamma_R^\infty = (-\infty, -R) \cup (R, \infty) \,.$$

Here we choose ϵ and so that there are no resonances on Γ. We also put

$$\Omega_A := \{\lambda \ : \ \mathrm{Im}\lambda \ge -A - \epsilon - \delta \log(1 + |\mathrm{Re}\lambda|)\} \,.$$

and define

$$\Pi_A(t) := i \sum_{\lambda \in \Omega_A} \mathrm{Res}_{\lambda = \lambda_j}(\rho R_V(\lambda)\rho) \,.$$

Hence

$$\rho U(t)\rho = \Pi_A(t) + E_{\Gamma_R}(t) + E_{\gamma_R}(t) + E_{\gamma_R^\infty}(t) \,, \qquad (2.31)$$

where (with obvious orientatations)

$$E_\gamma(t) := \frac{1}{2\pi}\int_\gamma e^{-it\lambda}(R_V(\lambda)) - R_V(-\lambda))w_1 d\lambda \,.$$

4. Let us assume that $w_1 \in H^2$, $\mathrm{supp} w_1 \subset [-K, K]$, $\rho \equiv 1$ on $[-K, K]$. For such w_1 we will show that

$$\|E_{\gamma_R}(t) w_1\|_{H^1}\,,\ \|E_{\gamma_R^\infty}(t) w_1\|_{H^1} \longrightarrow 0\,, \quad R \to \infty\,, \tag{2.32}$$

For that we note that

$$\begin{aligned} \rho(R_V(\lambda) - R_V(-\lambda))\rho w_1 &= \rho(R_V(\lambda) - R_V(-\lambda)) w_1 \\ &= \rho(R_V(\lambda) - R_V(-\lambda))(1 + \lambda^2)^{-1}(1 + D_x^2 + V) w_1\,, \end{aligned}$$

since

$$(R_V(\lambda) - R_V(-\lambda))(D_x^2 + V) = \lambda^2 (R_V(\lambda) - R_V(-\lambda))\,,$$

Using (2.28) we thus obtain

$$\|E_{\gamma_R^\infty}(t) w_1\|_{H^1} \leq C \int_R^\infty (1 + |\lambda|^2)^{-1} \|w_1\|_{H^2} \leq \frac{C}{R} \|w_1\|_{H^2}\,,$$

and

$$\|E_{\gamma_R}(t) w_1\|_{H^1} \leq \frac{C}{1 + R^2} \|w_1\|_{H2}\,.$$

Hence (2.32) holds for $w_1 \in H^2$.

5. We now return to (2.31) and see that

$$\rho U(t) \rho w_1 = \Pi_A(t) w_1 + E_\Gamma(t) w_1\,, \quad w_1 \in H^2\,.\,. \tag{2.33}$$

We will now show that for $t \gg 1$

$$\|E_\Gamma(t) w_1\|_{H^2} \leq C e^{-tA} \|w_1\|_{L^2}\,. \tag{2.34}$$

For that we again use (2.28) for $|\lambda| > R$ and the assumption that there are no poles of $R_V(\lambda)$ near Γ in a compact set. Thus we obtain:

$$\begin{aligned} \|E_\Gamma(t) w_1\|_{H^2} &\leq C e^{-At} \int_{\mathbb{R}} e^{-t\delta \log(1+|\lambda|} e^{-\delta T \log(1+|\lambda|)} (1 + |\lambda|) \|w_1\|_{L^2} \\ &\leq C e^{-At} \int_{\mathbb{R}} (1 + |\lambda|)^{-\delta(t-T)+1} \|w_1\|_{L^2} d\lambda \\ &\leq C' e^{-At} \|w_1\|_{L^2}\,, \quad t > T + 3/\delta\,. \end{aligned}$$

Since H^2 is dense in L^2 the decomposition (2.33) is valid for $w_1 \in L^2$ proving theorem. $\square$

2.3. **Scattering matrix in dimension one.**

Outside of the support of V, a solution of

$$(P_V - \lambda^2)u = 0 \tag{2.35}$$

can be written as a sum of an outgoing and incoming terms

$$u(x) = u_{\rm in}(x) + u_{\rm out}(x)\,, \quad |x| \geq R\,.$$

Following the conventions described in the beginning of this chapter,

$$u_{\rm in}(x) = b_{{\rm sgn}(x)} e^{-i\lambda|x|}\,, \quad u_{\rm out}(x) = a_{{\rm sgn}(x)} e^{i\lambda|x|}\,, \quad |x| \geq R\,.$$

In scattering we compare the incoming waves with the outgoing ones and mathematically that is captured by the scattering matrix which is defined as follows

$$S \;:\; \begin{pmatrix} b_- \\ b_+ \end{pmatrix} \longmapsto \begin{pmatrix} a_+ \\ a_- \end{pmatrix}\,. \tag{2.36}$$

To describe $S = S(\lambda)$ at frequency λ we need to find solutions to (2.35) of the following form:

$$u^\pm(x) = e^{\pm i\lambda x} + v^\pm(x, \lambda) \tag{2.37}$$

where $v^\pm(x,\lambda)$ is outgoing. It is easily found using the outgoing resolvent $R_V(\lambda)$:

$$v^\pm(x,\lambda) = -R_V(\lambda)\left(V e^{\pm i\lambda x}\right)\,.$$

This is well defined away from the poles of $R_V(\lambda)$. In the self-adjoint case that means that $u_\pm$ exist for $\lambda \in \mathbb{R} \setminus 0$.

To describe $S = S(\lambda)$ given by (2.36) we want to find epxressions of

$$v^\pm_{{\rm sgn}(x)}(\lambda) := e^{-i\lambda|x|} v^\pm(x,\lambda)\,, \quad |x| > R\,.$$

In terms of $\alpha_\pm$ and $\beta_\pm$ we see that

$$\begin{gathered} S(\lambda) : \begin{pmatrix} 1 \\ 0 \end{pmatrix} \longmapsto \begin{pmatrix} 1 + v^+_+(\lambda) \\ v^+_-(\lambda) \end{pmatrix}\,, \\ S(\lambda) : \begin{pmatrix} 0 \\ 1 \end{pmatrix} \longmapsto \begin{pmatrix} v^-_+(\lambda) \\ 1 + v^-_-(\lambda) \end{pmatrix}\,, \end{gathered} \tag{2.38}$$

which means that

$$S(\lambda) = I + A(\lambda)\,, \quad A(\lambda) = \begin{pmatrix} v^+_+(\lambda) & v^-_+(\lambda) \\ & \\ v^+_-(\lambda) & v^-_-(\lambda) \end{pmatrix}\,. \tag{2.39}$$

THEOREM 2.7 (**Scattering matrix in terms of the resolvent).** *The coefficients of $A(\lambda)$ are meromorphic functions of λ given by the following formulæ:*

$$v_\theta^\omega(\lambda) = \frac{1}{2i\lambda}\int_{\mathbb{R}} e^{i(\omega-\theta)\lambda x} V(x)(1 - e^{-i\omega x\lambda} R_V(\lambda)(e^{i\omega\lambda \bullet} V)(x))dx\,, \tag{2.40}$$

where $\theta, \omega \in \{\pm\}$.

Proof. We write

$$v_\theta^\omega(\lambda) = -e^{-i\lambda\theta x} R_0(\lambda)(I - VR_V(\lambda))(Ve^{i\omega\lambda\bullet})\,, \quad \theta x > R\,.$$

Using the explicit formula for $R_0(\lambda)$ we then notice that for f with $\mathrm{supp} f \subset [-R, R]$,

$$R_0(\lambda)f(x) = -\frac{1}{2i\lambda} e^{i\theta\lambda x}\int_{\mathbb{R}} e^{-i\theta\lambda y} f(y)dy\,, \quad \theta x \geq R\,.$$

Combining the two expressions we obtain (2.40). □

INTERPRETATION.

1. The coefficients $v_\theta^\omega(\lambda)$ have important physical interpretations:

$$\begin{aligned} T(\lambda) &= 1 + v_\pm^\pm(\lambda) \ \text{ is the transmission coefficient,} \\ R_+(\lambda) &= v_+^-(\lambda) \ \text{ is the right reflection coefficient,} \\ R_-(\lambda) &= v_+^-(\lambda) \ \text{ is the left reflection coefficient.} \end{aligned} \tag{2.41}$$

This can be seen from comparing (2.36) and (2.38). In (2.41) we have implicitely asserted that $v_+^+(\lambda) = v_-^-(\lambda)$. This can be seen by comparing the Wronskian, which are constant, for $\lambda \neq 0$:

$$W(u_+, u_-) := \begin{vmatrix} u_+ & u_- \\ \partial_x u_+ & \partial_x u_- \end{vmatrix} = \begin{cases} -2i\lambda(1 + v_-^-),, & x < -R\,, \\ -2i\lambda(1 + v_+^+)\,, & x > R\,. \end{cases}$$

The equality of v_-^- and v_+^+ can also be seen from (2.40) where we make a change $x \mapsto -x$ in the integral.

2. When V is real and $\lambda \in \mathbb{R} \setminus \{0\}$ then we can also take Wronskians of u_+ and $\bar{u}_-$. The important consequence is the *unitarity of the scattering matrix*: $S(\lambda)^* = S(\lambda)^{-1}$. A meromorphic continuation of this equality gives

$$V \in L^\infty_{\text{comp}}(\mathbb{R}; \mathbb{R}) \implies S(\bar{\lambda})^* = S(\lambda)^{-1}\,, \quad \lambda \in \mathbb{C}\,. \tag{2.42}$$

REMARK. We should think of $\pm$ as the element of the "sphere", $\mathbb{S}^0$, in one dimensional space. As we will see the same formula is valid in dimension n with $\theta, \omega \in \mathbb{S}^{n-1}$. The scattering "matrix" is then given as the sum of the idenity and an operator defined by an integral kernel (2.40) in $\mathbb{S}^{n-1} \times \mathbb{S}^{n-1}$. Of course the interpretation of reflected and transmited waves is then less clear.

The representation given in Theorem 2.7 gives us important estimates for the coefficients of the scattering matrix in the physical half plane, $\mathrm{Im}\lambda \geq 0$:

THEOREM 2.8 (Estimates on the scattering matrix). *For*

$$\mathrm{Im}\lambda \geq 0\,, \quad |\lambda| \geq C_0\,,$$

we have

$$\|e^{\mp i\lambda x} R_V(\lambda)(V e^{\pm i\lambda \bullet})\|_\infty \leq \frac{C_1}{|\lambda|}\,. \tag{2.43}$$

Consequently, (2.40) *implies that for* $\mathrm{Im}\lambda \geq 0$*,* $|\lambda| \geq C_0$*,*

$$v_\pm^\pm(\lambda) = \frac{i}{2\lambda}\left(\widehat{V}(0) + \mathcal{O}(1/|\lambda|)\right)\,. \tag{2.44}$$

Proof. 1. The estimate (2.28) shows that for $\mathrm{Im}\lambda \geq 0$, $|\lambda| \geq C_0$ (in fact, under our assumptions, in a large region),

$$\lambda \mapsto f_\pm(x,\lambda) := e^{\mp i\lambda x} R_V(\lambda)(V e^{\pm i\lambda \bullet})(x)\,, \tag{2.45}$$

is holomorphic. If C_0 is large enough (2.9) shows that the Neumann series of $(I + \rho R_0(\lambda) V)^{-1}$ converges as operator $L^2 \to L^2$.

Hence

$$f_\pm(x,\lambda) := \left(R_0^\omega(\lambda)\rho(I + V R_0^\omega(\lambda)\rho)^{-1} V\right)(x)\,,$$

$$R_0^\omega(\lambda)(x,y) := e^{\mp i\lambda x} R_0(\lambda)(x,y) e^{\pm i\lambda y} = \frac{i}{2\lambda} e^{i\lambda(|x-y| \mp (x-y))}\,,$$

where in the last line we defined, and calculated, the Schwartz kernel of $R_0^\omega(\lambda)$.

2. We see that for $\mathrm{Im}\lambda \geq 0$ we still have

$$\|V R_0^\omega(\lambda)\rho\| \leq C/|\lambda|$$

and hence the Neumann series for $(I + V R_0^\omega(\lambda)\rho)^{-1}$ converges. Similarly,

$$R_0^\omega(\lambda)\rho = \mathcal{O}(1/|\lambda|) : L^2 \to L^\infty\,, \quad \mathrm{Im}\lambda \geq 0\,,$$

which concludes the proof. $\square$

REMARK. We should stress that unlike many reasults in this chapter the statements about the scattering matrix for $\mathrm{Im}\lambda \geq 0$ remain valid for real potential satisfying very weak decay conditions – see [Mel] for one account of that and for references.

The determinant of the scattering matrix is related to the determinant defined by (2.20):

THEOREM 2.9 (A determinant identity). *Suppose that $V \in L^\infty_{\mathrm{comp}}(\mathbb{R};\mathbb{C})$. For $\rho \in L^\infty_{\mathrm{comp}}$, $\rho V = V$, let*

$$D(\lambda) := \det(I + VR_0(\lambda)\rho)\,.$$

Then

$$\frac{D(-\lambda)}{D(\lambda)} = \det S(\lambda)\,, \tag{2.46}$$

where $S(\lambda)$ is the scattering matrix.

Proof. 1. We first write

$$\rho(R_0(\lambda) - R_0(-\lambda))\rho = \frac{i}{2\lambda} E(\bar\lambda)^* E(\lambda)\,, \tag{2.47}$$

where $E(\lambda) : L^2(\mathbb{R}) \longrightarrow \mathbb{C}^2$,

$$E(\lambda)u := \left(\int_{\mathbb{R}} e^{i\lambda x} u(x)\rho(x)dx, \int_{\mathbb{R}} e^{-i\lambda x} u(x)\rho(x)dx \right)\,.$$

In other words,

$$E(\bar\lambda)^* E(\lambda) = \rho(x)e^{i\lambda x} \otimes \rho(y)e^{-i\lambda y} + \rho(x)e^{-i\lambda x} \otimes \rho(y)e^{i\lambda y}\,. \tag{2.48}$$

2. Now,

$$(I + VR_0(-\lambda)\rho) = \\ (I + VR_0(\lambda)\rho)(I - (I + VR_0(\lambda)\rho)^{-1}(iVE(\bar\lambda)^*E(\lambda)/2\lambda))\,,$$

and we need to show that

$$\det{}_{\mathbb{C}^2} S(\lambda) = \det{}_{L^2}(I + T(\lambda))\,, \tag{2.49}$$

where $T(\lambda)$ is the rank two operator appearing in the expression above:

$$T(\lambda) := \frac{1}{2i\lambda}(I + VR_0(\lambda)\rho)^{-1} VE(\bar\lambda)^* E(\lambda)\,.$$

Since

$$\det_{L^2}\left(I+\sum_{\ell,k=1}^{K}\varphi_k\otimes\psi_\ell\right)=\det_{\mathbb{C}^K}\left(I_{\mathbb{C}^k}+A\right)\,,$$
$$A_{k\ell}:=\int\varphi_k(x)\psi_\ell(x)dx\,,$$

identity (2.49) follows from calculating φ_k, ψ_ℓ $1\leq k,\ell\leq 2$ for $T(\lambda)$ and comparing the answer with (2.40). □

A multiplicity of a pole of $S(\lambda)$ and $S(\lambda)^{-1}$ is defined using the determinant of the scattering matrix. The poles of the scattering matrix are sometimes called **scattering poles**. Theorem 2.9 combined with Theorem 2.4 gives

THEOREM 2.10 (Multiplicities of scattering poles I). *The multiplicity of a scattering pole defined by*

$$m_S(\lambda)=\frac{1}{2\pi i}tr\oint S(\zeta)^{-1}\partial_\zeta S(\zeta)d\zeta\,, \tag{2.50}$$

where the integral is over a positively oriented circle which includes λ and no other pole or zero of $\det S(\lambda)$*, is related to the multiplicity of a scattering resonance* (2.13) *as follows:*

$$m_S(\lambda)=m_R(\lambda)-m_R(-\lambda)\,. \tag{2.51}$$

2.4. **Asymptotics for the counting function.**

In this section we will prove a Weyl law for the number of scattering resonances of a compactly supported, bounded, complex valued potential. In higher dimensions only weaker results are known and for the existence of resonances we need to assume that the potential is real valued: as we will see in Chapter 3 a complex valued compactly supported potential in three dimensions may have no resonances at all.

THEOREM 2.11 (Asymptotics for the number of resonances). *Suppose that $V\in L^\infty_{\rm com}(\mathbb{R};\mathbb{C})$. Then*

$$\sum\{m_R(\lambda):|\lambda|\leq r\,,\pm\mathrm{Re}\lambda\geq 0\}=\frac{|\mathrm{chsupp}V|}{\pi}r(1+o(1))\,, \tag{2.52}$$

as $r\longrightarrow\infty$. Here chsupp *is the convex hull of the support.*

In addition, for any $\epsilon > 0$,

$$\sum \{ m_R(\lambda) : |\lambda| \leq r \,, |\mathrm{Im}\lambda| \geq \epsilon |\mathrm{Re}\lambda| \} = o(r) \,,$$

as $r \longrightarrow \infty$.

Before proving the theorem we need some estimates for the determinant $D(\lambda) = \det(I + VR_0(\lambda)\rho)$. These estimates will also be useful in the section on trace formulæ.

THEOREM 2.12 (Determinant estimates). *The determinant* $D(\lambda)$ *defined by* (2.20) *satisfies*

(2.53)

$$\lim_{t \to +\infty} D(e^{i\theta} t) = 1 \,, \quad 0 < \theta < \pi \,, \quad |D(\lambda)| \leq C_1 (1 + 1/|\lambda|) \,, \quad \mathrm{Im}\lambda \geq 0 \,,$$

$$|\lambda D(\lambda)| \leq C_2 \exp(\tau |\lambda|) \,, \quad \lambda \in \mathbb{C} \,, \quad \tau := |\mathrm{chsupp}\, V| \,,$$

where chsupp V *is the convex hull of the support of* V.

We start with the following lemma concerning trace class norms of the free cut-off resolvent;

LEMMA 2.13. *Suppose that* $\rho \in L^\infty(\mathbb{R})$ *and* supp$\rho \subset [-L, L]$. *Then*

$$\|\rho R_0(\lambda) \rho\|_{\mathcal{L}_1} \leq \frac{C \exp(2L(\mathrm{Im}\lambda)_-)}{|\mathrm{Im}\lambda|} \,, \quad \mathrm{Im}\lambda \neq 0 \,,$$

$$\|\rho R_0(\lambda) \rho\|_{\mathcal{L}_1} \leq C + \frac{C}{|\lambda|} \,, \quad \lambda \in \mathbb{R} \,. \tag{2.54}$$

Proof. 1. We start with the case of $\mathrm{Im}\lambda > 0$. In that case, as operators on L^2,

$$R_0(\lambda) = (D_x^2 - \lambda^2)^{-1} = (D_x - \lambda)^{-1} (D_x + \lambda)^{-1} \,.$$

Using the explicit formulae for the Schwartz kernel, n

$$(D_x \pm \lambda)^{-1}(x, y) = e^{\pm i \lambda (x - y)} (x - y)_\pm^0 \,, \quad \mathrm{Im}\lambda > 0 \,,$$

we see that

$$\int_{\mathbb{R}} \int_{\mathbb{R}} |\rho(x) (D_x - \lambda)^{-1}(x, y)|^2 dx \leq \frac{2L \|\rho\|_\infty}{\mathrm{Im}\lambda} \,,$$

and similarly for $(D_x + \lambda)^{-1} \rho$. Hence

$$\|\rho R_0(\lambda) \rho\|_{\mathcal{L}_1}^2 \leq \|\rho (D_x + \lambda)^{-1}\|_{\mathcal{L}_2} \|(D_x - \lambda)^{-1} \rho\|_{\mathcal{L}_2} \leq \frac{C_\rho^2}{\mathrm{Im}\lambda^2} \,.$$

2. To prove the estimate for $\mathrm{Im}\lambda \leq$ we use (2.47), (2.48) and the fact that
$$\|u \otimes v\|_{\mathcal{L}_1} = \|u\|_{L^2}\|v\|_{L^2} .$$
This gives,
$$\begin{aligned}\|\rho R_0(\lambda)\rho\|_{\mathcal{L}_1} &\leq \|\rho R_0(-\lambda)\rho\|_{\mathcal{L}_1} + \frac{1}{|\lambda|}\|\rho e^{i\lambda\bullet}\|_{L^2}\|\rho e^{-i\lambda\bullet}\|_{L^2} \\ &\leq \frac{C_\rho}{|\mathrm{Im}\lambda|} + \frac{C_\rho e^{-2L\mathrm{Im}\lambda}}{|\lambda|} \\ &\leq \frac{2C_\rho e^{2L(\mathrm{Im}\lambda)_-}}{|\mathrm{Im}\lambda|} .\end{aligned}$$

3. The second inequality in (2.54) follows from the bounds on the norm of
$$\rho R_0(\lambda)\rho : L^2 \longrightarrow H^1 ,$$
given in Theorem 2.1. $\square$

Proof of Theorem 2.12. 1. To study $D(e^{i\theta}t)$ we use (B.19) with $A = VR_0(\lambda)\rho$ and $B = 0$:
$$|D(e^{i\theta}t) - 1| \leq \|V\|_\infty\|\rho R_0(\lambda\rho\|_{\mathcal{L}_1}e^{1+\|V\|_\infty\|\rho R_0(\lambda)\rho\|_{\mathcal{L}_1}}$$
The first estimate in (2.54) shows that the right hand side goes to 0 as $t \to +\infty$ for $0 < \theta < \pi$.

2. The same argument using the second estimate in (2.54) gives the bound
$$D(\lambda) = \mathcal{O}(1) + \mathcal{O}(1/|\lambda|) , \quad \mathrm{Im}\lambda \geq 0 .$$

3. To obtain estimates in $\mathrm{Im}\lambda \leq 0$ we use Theorem 2.9:
$$D(\lambda) = \det S(-\lambda)D(-\lambda) .$$
Hence we need to estimate $\det S(-\lambda)$ for $\mathrm{Im}\lambda \leq 0$ and $|\lambda| \geq 0$,

Theorem 2.8 shows that for $\mathrm{Im}\lambda \leq 0$, $|\lambda| \geq C_0$, we have
$$|\det S(-\lambda)| = 1 + v_+^-(-\lambda)v_-^+(-\lambda) + \mathcal{O}(1/|\lambda|) . \tag{2.55}$$
The estimate for $\lambda D(\lambda)$ follows from (2.40) and (2.43) and from estimates established for $\mathrm{Im}\lambda \geq 0$. $\square$

Proof of Theorem 2.11. 1. Using Theorem 2.4 we will prove the theorem by obtaining an asymptotic formula for the number of zeros of the entire function
$$f(\lambda) := \lambda D(\lambda) ,$$

where $D(\lambda)$ is defined in (2.20). The factor λ removes the pole at $\lambda = 0$ – see the second estimate in (2.53). We note that we do need any additional information about the multiplicity at $\lambda = 0$ for the asymptotic statement.

2. By rescaling and translation we can assume that

$$\operatorname{chsupp} V = [-1, 1] .$$

In view of Theorem D.1 it suffices to show that

$$\int_{\mathbb{R}} \frac{\log^+ f(x)}{1+x^2} dx < \infty , \qquad I_f = [0, -4i] , \tag{2.56}$$

where I_f is the indicator diagram of f. The first condition in (2.56) follows immediately from the second inequality in (2.53).

3. In view of (D.10) to establish $I_f = [0, -4i]$ all we need is to calculate the precise type of $D(\lambda)$. The upper bound is already provided in the last inequality in (2.53).

For $\mathrm{Im}\lambda < 0$ we use Theorem 2.9 and (2.55) to see that

$$D(\lambda) = v_+^-(-\lambda) v_-^+(-\lambda) + \mathcal{O}(1) , \quad \mathrm{Im}\lambda \leq 0 , \quad |\lambda| \geq C .$$

The type of $D(\lambda)$ will be exactly 4 if we show that we *cannot* have

$$|v_\pm^\mp(-\lambda)| \leq C e^{2(1-\delta)|\lambda|} , \quad \mathrm{Im}\lambda \leq 0 , \quad |\lambda| \geq C , \tag{2.57}$$

with $\delta > 0$ for *either* $\pm$.

4. Let us choose $\beta > 0$ such that $R_V(-\lambda)$ is holomorphic for $\mathrm{Im}\lambda \leq -\beta$ – that is possible as there are only finitely many poles on $R_V(-\lambda)$ in $\mathrm{Im}\lambda \leq 0$. Hence $f_(x, -\lambda)$ has no poles in $\mathrm{Im}\lambda \leq -\beta$ and Theorem 2.8 shows that $|f_-(x, -\lambda)| \leq C/|\lambda|$ there.

To show that (2.57) cannot hold for v_+^- we use (2.40) and the notation (2.45) to write

$$v_-^+(-\lambda - i\beta) = \frac{i}{2(\lambda+\beta)} \int_{\mathbb{R}} e^{2i\lambda x} V(x) e^{-2\beta x} (1 - f_+(x, -\lambda - i\beta)) dx .$$

Since v_-^+ is bounded for $-\beta \leq \mathrm{Im}\lambda \leq 0$, $|\lambda| \geq C_0$, (2.57) implies that

$$|v_-^+(-\lambda - i\beta)| \leq C e^{2(1-\delta)|\lambda|} , \quad \mathrm{Im}\lambda \leq 0 , \quad |\lambda| \geq C , \tag{2.58}$$

with $\delta > 0$, and we need to find a contradiction to these statement. To simplify notation let us put

$$\begin{aligned} V_\epsilon^\beta(x) &:= \mathbb{1}_{[1-\epsilon,1]} V(x) e^{-2\beta x} , \\ g_\epsilon^\beta(x, \lambda) &= \mathbb{1}_{[1-\epsilon,1]} f_+(x, -\lambda - i\beta) . \end{aligned} \tag{2.59}$$

5. Take $\epsilon < \delta$, and define

$$I_\epsilon(\lambda) := \int_{\mathbb{R}} e^{2i\lambda x} V_\epsilon^\beta(x)(1 - g_\epsilon^\beta(x, \beta))dx \, .$$

which holomorphic in $\mathrm{Im}\lambda \leq 0$.

Since $\epsilon < \delta$ we have $|I_\epsilon(\lambda)| \leq Ce^{(2-\epsilon)|\lambda|}$ due to the assumption (2.58) and the fact that, for $\mathrm{Im}\lambda \leq 0$,

$$\int_{\mathbb{R}} e^{2i\lambda x} \mathbb{1}_{[-1,1-\epsilon]}(x) V(x) e^{-\beta x}(1 - f_-(x, -\lambda - i\beta))dx = \mathcal{O}(e^{2(1-\epsilon)|\lambda|}) \, ,$$

Paley-Wiener theor!em then shows that

$$\widehat{I}_\epsilon(x) = 0 \, , \quad x > 1 - \epsilon \, ,$$

that is

$$V_\epsilon^\beta(x) = \frac{1}{2\pi} \int_{\mathbb{R}} \int_{\mathbb{R}} e^{2i\lambda(y-x)} V_\epsilon^\beta(y) g_\epsilon^\beta(y, \lambda) d\lambda dy \, .$$

for $1 - \epsilon \leq x \leq 1$. Plancherel's theorem and the Cauchy Schwartz inequality then imply that

$$\begin{aligned} \|V_\epsilon^\beta\|_{L^2} &= (2\pi)^{-1} \left\| \int_{\mathbb{R}} e^{2i\lambda y} V_\epsilon^\beta(y) g_\epsilon^\beta(y, \lambda) dy \right\|_{L^2(d\lambda)} \\ &\leq (2\pi)^{-1} \left\| \|V_\epsilon^\beta\|_{L^2} \|g_\epsilon^\beta(y, \lambda)\|_{L^2(dy)} \right\|_{L^2(d\lambda)} \\ &= (2\pi)^{-1} \|V_\epsilon^\beta\|_{L^2} \|g_\epsilon^\beta\|_{L^2(dy,d\lambda)} \end{aligned} \tag{2.60}$$

We recall that $g_\epsilon^\beta \in L^2(d\lambda)$ because of the $\mathcal{O}(1/|\lambda|)$ decay of f_+ given in Theorem 2.8.

Because of the factor $\mathbb{1}_{[1-\epsilon,1]}$ in the definition of g_ϵ^β in (2.59), we have

$$(2\pi)^{-1} \|g_\epsilon^\beta\|_{L^2(dy,d\lambda)} \longrightarrow 0 \, , \quad \epsilon \longrightarrow 0+ \, .$$

It follows from (2.60) that for ϵ small enough $\|V_\epsilon^\beta\|_{L^2} = 0$. Recalling (2.59) this means that

$$V(x) = 0 \quad \text{for } 1 - \epsilon < x < 1,$$

contradicting the assumption that $\mathrm{chsupp}\,\mathrm{V} = [-1, 1]$.

6. The same argument applies in the case (2.57) holds for v_+^- that

$$V(x) = 0 \quad \text{for } -1 < x < 1 - \epsilon,$$

leading again to contradiction. Hence (2.56) holds and Theorem D.1 gives the asymptotics of resonances. $\square$

2.5. **Trace formulæ.**

We will now prove two trace formulas: one involving the scattering matrix and another relating resonances to the wave group.

THEOREM 2.14 (Birman-Krein formula I). *Suppose that* $V \in L^\infty_{\rm comp}(\mathbb{R};\mathbb{R})$.

Then for $f \in \mathscr{S}(\mathbb{R})$ *the operator* $f(P_V) - f(P)$ *is of trace class and*

$$\begin{aligned} \operatorname{tr}\left(f(P_V) - f(P_0)\right) = \frac{1}{2\pi i}\int_0^\infty f(\lambda^2)\operatorname{tr}\left(S(\lambda)^{-1}\partial_\lambda S(\lambda)\right)d\lambda \\ + \sum_{k=1}^K f(E_k) + \frac{1}{2}(m_R(0)-1)f(0)\,, \end{aligned} \tag{2.61}$$

where $S(\lambda)$ *is the scattering matrix and* $E_K < \cdots < E_1 < 0$ *are the (negative) eigenvalues of* P_V.

Theorem 2.14 is a consequence of the determinant identity presented in Theorem 2.9.

INTERPRETATION. As in the beginning of Section 2.2 we can compare this result to a result involving eigenvalues. Let us denote he Dirichlet realization of P_V on $[a,b]$ by P_V^D. The spectrum of P_V^D is discrete,

$$E_N < E_{N-1} < \cdots < E_1 < 0 < \lambda_0^2 < \lambda_1^2 < \cdots \to \infty\,.$$

For $f \in \mathscr{S}(\mathbb{R})$, we have

$$\operatorname{tr} f(P_V^D) = \sum_{j=0}^\infty f(\lambda_j^2) + \sum_{k=1}^N f(E_k) \tag{2.62}$$

which can be written as

$$\operatorname{tr} f(P_V^D) = \int_0^\infty f(\lambda^2)\frac{dN(\lambda)}{d\lambda}d\lambda + \sum_{k=1}^N f(E_k) \tag{2.63}$$

where

$$N(\lambda) = \#\{\lambda_j^2 : \lambda_j^2 \leq \lambda^2\}$$

is the counting function for the positive eigenvalues of P_V^D.

Hence we have the following correspondence between confined (discrete spectrum) and open (continuous spectrum/scattering) problems:

$$N(\lambda) \quad \longleftrightarrow \quad \frac{1}{2\pi i}\log S(\lambda)\,.$$

Since $S(\lambda)$ is unitary for $\lambda \in \mathbb{R}$ the right hand side is real. Since only the derivatives appear in the formula we can choose the branch of log arbitrarily.

Proof of Theorem 2.14. **Still need to sort out the multiplicity at 0 – the proof below works when $f(0) = 0$.** For simplicity we assume that there are *no* negative eigenvalues as their contribution is easy to analyse.

1. Since we assume that $V \in L^\infty(\mathbb{R};\mathbb{R})$, P_V is selfadjoint and we can use Stone's formula as we did in the proof of Theorem 2.5. That gives

$$f(P_V) = \frac{1}{2\pi i}\int_{\mathbb{R}} f(\lambda^2)(R_V(\lambda) - R_V(-\lambda))2\lambda d\lambda\,,$$

2. Consequently, using

$$\begin{aligned} R_V(\lambda) - R_0(\lambda) &= -R_V(\lambda)VR_0(\lambda) \\ &= -R_0(\lambda)(I + VR_0(\lambda)\rho)^{-1}VR_0(\lambda)\,, \end{aligned}$$

we obtain

$$f(P_V) - f(P_0) = \sum_{\pm} \frac{1}{2\pi i}\int_{\mathbb{R}} f(\lambda^2)B(\pm\lambda)d\lambda\,,$$

where

$$B(\lambda) := 2\lambda R_0(\lambda)(I + VR_0(\lambda)\rho)^{-1}VR_0(\lambda)\,. \tag{2.64}$$

This operator is of trace class for $\mathrm{Im}\lambda > 0$ and

$$\|B(\lambda)\|_{\mathcal{L}_1} \leq \frac{C}{|\mathrm{Im}\lambda|^2}\,.$$

We recall from (2.14) that there are no poles on the real axis – except for the possible pole at $\lambda = 0$.

Let $g \in \mathscr{S}(\mathbb{C})$, $\mathrm{supp} g \subset \{|\mathrm{Im}\lambda| \leq 1\}$, be an *almost analytic extention* of $f(\lambda^2)$:

$$g(\lambda) = f(\lambda^2)\,,\ \lambda \in \mathbb{R}\,,\ \ \partial_{\bar\lambda} g(\lambda) = \mathcal{O}(|\mathrm{Im}\lambda|^\infty)\,.$$

Green's formula then shows that

$$\begin{aligned} f(P_V) - f(P_0) &= \frac{1}{2\pi i}(t_+(f) + t_-(f))\,, \\ t_\pm(f) &:= 2i\int_{\pm\mathrm{Im}\lambda > 0} \partial_{\bar\lambda} g(\pm\lambda)B(\pm\lambda)d\mathcal{L}(\lambda)\,. \end{aligned} \tag{2.65}$$

We conclude that

$$\|t_\pm(f)\|_{\mathcal{L}_1} \leq \int_{0<\pm\mathrm{Im}\lambda<1} \mathcal{O}(|\mathrm{Im}\lambda|^\infty(1+|\lambda|)^{-\infty})|\mathrm{Im}\lambda|^{-2} d\mathcal{L}(\lambda) < \infty\,.$$

This proves the claim that

$$f(P_V) - f(P_0) \in \mathcal{L}_1\,. \tag{2.66}$$

3. To calculate the trace of $f(P_V) - f(P_0)$ we are going to use Theorem 2.9. Taking logarithmic derivatives of both sides of (2.46) we obtain

$$\begin{gathered} \mathrm{tr}F(-\lambda) + \mathrm{tr}F(\lambda) = \mathrm{tr}\,\partial_\lambda S(\lambda) S(\lambda)^{-1}\,, \\ F(\lambda) := \partial_\lambda (V R_0(\lambda)\rho)(I + V R_0(\lambda)\rho)^{-1}\,. \end{gathered} \tag{2.67}$$

We claim that for $\mathrm{Im}\lambda > 0$ we have

$$\mathrm{tr}F(\lambda) = \mathrm{tr}B(\lambda)\,, \tag{2.68}$$

where $B(\lambda)$ was defined by (2.64).

To see (2.68) we use the fact that $R_0(\lambda)$ is bounded on L^2 for $\mathrm{Im}\lambda > 0$ and hence

$$\partial_\lambda (V R_0(\lambda)\rho) = 2\lambda V R_0(\lambda)^2 \rho\,.$$

Using this, the cyclicity of the trace, and $\rho V = V$, we obtain, always for $\mathrm{Im}\lambda > 0$,

$$\mathrm{tr}F(\lambda) = 2\lambda \mathrm{tr} R_0(\lambda)(I + V R_0(\lambda)\rho)^{-1} V R_0(\lambda) = \mathrm{tr}B(\lambda)\,,$$

which is (2.68).

4. We combine (2.65), (2.68) and (2.67) to obtain, under the assumption that there is no discrete spectrum,

$$\mathrm{tr}\,(f(P_V) - f(P_0)) = \frac{1}{4\pi i} \int_0^\infty f(\lambda^2) \mathrm{tr}\left(S(\lambda)^{-1}\partial_\lambda S(\lambda)\right)\,.$$

Since $\det S(-\lambda) = (\det S(\lambda))^{-1}$ (see for instance (2.46)) the integrand is even which gives (2.61). □

As a consequence of Theorem 2.14 we have the following *trace formula for resonances*:

THEOREM 2.15 (Trace formula for resonances I). *Suppose that* $V \in L^\infty_{\mathrm{comp}}(\mathbb{R};\mathbb{R})$. *Then for* $\varphi \in C^\infty_{\mathrm{c}}(\mathbb{R})$ *the operator*

$$\int_{\mathbb{R}} \varphi(t) \left(\cos t\sqrt{P}_V - \cos t\sqrt{P}_0\right) dt$$

is of trace class, and, in the sense of distributions on $\mathbb{R} \setminus \{0\}$*,*

$$
\begin{aligned}
2\mathrm{tr}\left(\cos t\sqrt{P_V} - \cos t\sqrt{P_0}\right) &= \sum_{\mathrm{Im}\lambda<0} m_R(\lambda) e^{-i\lambda|t|} \\
&+ \sum_{k=1}^{K} \cosh t\sqrt{-E_k} + m_R(0) - 1\,.
\end{aligned} \tag{2.69}
$$

INTERPRETATION. The expansion (2.24) leads directlty to a trace formula for, say, the Dirichlet realization of P_V on $[a, b]$. As before we that Dirichlet realization by P_V^D. Assuming for simplicity that there are no non-positive eigenvalues we have

$$
2\,\mathrm{tr}\,\cos t\sqrt{P_V^D} = \sum_{\lambda^2 \in \mathrm{Spec}(P_V^D)} e^{-i\lambda t}\,.
$$

Hence the expansion (2.69) is an exact analogue of this well known consequence of the spectral theorem. What is remarkable is the fact that unlike the resonance wave expansions given in Theorem 2.5 the trace formula (2.69) is *exact*.

Proof of Theorem 2.15. We again make the simplifying assumption that there are no eigenvalues.
1. We observe that both sides of (2.69) are even in t. Hence (2.69) is equivalent to the following statement: for $\varphi \in C_c^\infty((0, \infty))$

$$
\begin{aligned}
f(P_V) - f(P_0) &= \sum_{\mathrm{Im}\lambda<0} \widehat{\varphi}(\lambda) m_R(\lambda) + (m_R(0) - 1)\widehat{\varphi}(0)\,, \\
f(z) &:= \widehat{\varphi}(\sqrt{z}) + \widehat{\varphi}(-\sqrt{z})\,, \quad f \in \mathscr{S}(\mathbb{R})\,.
\end{aligned} \tag{2.70}
$$

By Theorem2.14, and because

$$
\partial_\lambda(\log\det S(\lambda)) = \partial_\lambda(\log\det S(-\lambda))\,,
$$

we need to show that

$$
\frac{1}{2\pi i}\int_{-\infty}^{\infty} \widehat{\varphi}(\lambda)\partial_\lambda(\log\det S(\lambda))d\lambda = \sum_{\mathrm{Im}\lambda<0} \widehat{\varphi}(\lambda) m_R(\lambda)\,. \tag{2.71}
$$

2. Recall that

$$
\det S(\lambda) = \frac{D(-\lambda)}{D(\lambda)}\,,
$$

and that $\lambda D(\lambda)$ is an entire function of exponential type. Then using the Hadamard factorization theorem, we have

$$D(\lambda) = \lambda^{m_R(0)-1} e^{a_0 + a_1 \lambda} P(\lambda) \,, \quad P(\lambda) := \prod_{\mathrm{Im}\mu < 0} E_1(\lambda/\mu)^{m_R(\mu)} \,,$$

where $E_1(z) := (1-z)e^z$. We observe that

$$\partial_\lambda^2 \log P(\lambda) = - \sum_{\mathrm{Im}\mu<0} \frac{m_R(\mu)}{(\lambda - \mu)^2} \,,$$

and hence, using (2.46), we have

$$\partial_\lambda^2 (\log \det S(\lambda)) = \sum_{\mathrm{Im}\mu<0} \frac{m_R(\mu)}{(\lambda-\mu)^2} - \sum_{\mathrm{Im}\mu<0} \frac{m_R(\mu)}{(\lambda+\mu)^2} \,.$$

3. Define $g \in \mathcal{S}$ by $g'(\lambda) = \widehat{\varphi}(\lambda)$. Note that such g exists and is unique as $0 = \varphi(0) = (2\pi)^{-1} \int \varphi$. Then,

$$\begin{aligned} \int_{\mathbb{R}} \widehat{\varphi}(\lambda) \partial_\lambda (\log \det S(\lambda)) d\lambda &= - \int_{\mathbb{R}} g(\lambda) \partial_\lambda^2 (\log \det S(\lambda)) d\lambda \\ &= \sum_{\pm} \pm \sum_{\mathrm{Im}\mu<0} \int_{\mathbb{R}} \frac{m_R(\mu)}{(\lambda \mp \mu)^2} g(\lambda) d\lambda \\ &= 2\pi i \sum_{\mathrm{Im}\mu<0} m_R(\mu) g'(\mu) \\ &= 2\pi i \sum_{\mathrm{Im}\mu<0} m_R(\mu) \widehat{\varphi}(\mu) \,, \end{aligned}$$

where we deformed the contour in the integral using the fact that

$$\widehat{\varphi} \in C^\infty_{\mathrm{c}}((0,\infty)) \implies |g(\lambda)| = \mathcal{O}((1+|\lambda|)^{-\infty}) \,, \quad \text{for } \mathrm{Im}\lambda \leq 0,$$

Since this gives (2.71) the proof is complete. $\square$

In Chapter 4 we will obtain a general version of Theorem 2.15. In dimension one we have however a more precise version unavailable in higher dimensions:

THEOREM 2.16 (Improved trace formula). *Suppose that $V \in L^\infty_{\rm com}$. Then, in the sense of distrubutions on $\mathbb{R}$*

(2.72)
$$
\begin{gathered}
2\mathrm{tr}\left(\cos t\sqrt{P_V} - \cos t\sqrt{P_0}\right) = 4|\mathrm{chsupp}\, V|\delta_0(t) + \frac{1}{2}(m_R(0)-1) \\
+ \mathrm{p.v.}\sum_{\mathrm{Im}\lambda<0} m_R(\lambda)e^{-i\lambda t} + \sum_{k=1}^{K} \cosh t\sqrt{-E_k}\,,
\end{gathered}
$$

where for $\varphi \in C^\infty_{\rm c}(\mathbb{R})$

$$
\left\langle \mathrm{p.v.}\sum_{\mathrm{Im}\lambda<0} m_R(\lambda)e^{-i\lambda t}, \varphi \right\rangle := \lim_{\Lambda\to\infty} \sum_{\substack{\mathrm{Im}\lambda<0 \\ |\lambda|\leq\Lambda}} m_R(\lambda)\widehat{\varphi}(\lambda)\,.
$$

2.6. **Complex scaling in one dimension.**

In this section we present the simplest case of the method of *complex scaling* which produces a natural family of non-selfadjoint operators whose discrete spectrum consists of resonances.

The idea is to consider D_x^2 as a restriction of the complex second derivative D_z^2 to the real axis thought of as a contour in $\mathbb{C}$. This contour is then deformed away from the support of V so that P can be restricted to it. This provides ellipticity at infinity at the price of losing self-adjointness.

An account of this method in higher dimensions and of its relation to the method of perfectly matched layers (PML) [Be] will be provided in Chapter 5. Again, in one dimension we can provide a low-tech self-contained presentation.

Let $\Gamma \subset \mathbb{C}$ be a curve which is a graph over $\mathbb{R}$ of a function $g : \mathbb{R} \to \mathbb{R}$ satisfying the following conditions:

$$
\begin{gathered}
g \in C^1(\mathbb{R};\mathbb{R})\,, \quad g' \geq 0\,, \quad g([-R,R]) = 0\,, \quad \text{where } \mathrm{supp}(V) \subset (-R,R), \\
\Gamma := \{x + ig(x) \;:\; x \in \mathbb{R}\} \subset \mathbb{C}\,.
\end{gathered}
$$

We now define differentiation and integration of functions mapping Γ to $\mathbb{C}$. Let $\gamma(t)$ be a parametrization $\mathbb{R} \to \Gamma$, and let $f \in C^1(\Gamma)$ in the sense that $f \circ \gamma \in C^1(\mathbb{R})$. We define

$$
\partial_z^\Gamma f(z_0) = \gamma'(t_0)^{-1}\partial_t(f\circ\gamma)(t_0)
$$

where $\gamma(t_0) = z_0$ (the inverse and the multiplication are in the sense of complex numbers), and further define

$$D_z = -i\partial_z^\Gamma .$$

By the chain rule, if f is differentiable in a neighborhood of Γ, this is independent of parametrization. In fact, if

$$\gamma(t) = \gamma_1(t) + i\gamma_2(t) , \quad \gamma_j : \mathbb{R} \longrightarrow \mathbb{R} ,$$

then

$$\gamma'(t_0)^{-1}\partial_t(f\circ\gamma)(t_0) = \gamma'(t_0)^{-1}(\partial_x f(z_0)\gamma_1'(t_0) + \partial_y f(z_0)\gamma_2'(t_0))$$

If α is another parametrization, $\alpha(s_0) = z_0$, then

$$\gamma'(t_0) = c(s_0)\alpha'(s_0) \ \text{ for some factor } c(s_0) \in \mathbb{R},$$

as two tangent vectors must be parallel. This factor cancels in the expression above, guaranteeing independence of parametrization.

Moreover, if f is holomorphic in a neighborhood of Γ, then the Cauchy-Riemann equation, $\partial_y f = i\partial_x f$, shows that

$$\partial_z^\Gamma f = \partial_x f = \partial_z f ,$$

so in this case ∂_z^Γ coincides with the holomorphic differential operator.

To integrate along the curve we can use both the complex contour measure and the arclength measure, denoted

$$dz = \gamma'(t)dt , \quad d|z| = |\gamma'(t)|dt ,$$

respectively. The space $L^2(\Gamma)$ is defined using the second measure.

Note that V is a well defined function on Γ, so that putting

$$P_\Gamma := (D_z^\Gamma)^2 + V(z) ,$$

makes sense.

For the main theorem we introduce two angles as follows. Fix $0 \leq \theta_1 \leq \theta_2 \leq \pi/2$ and $\varepsilon > 0$ having the property that for every $z \in \Gamma$ outside of some compact set, either $\theta_1 + \varepsilon \leq \arg z \leq \theta_2 - \varepsilon$ or $\theta_1 + \varepsilon \leq \arg(-z) \leq \theta_2 - \varepsilon$. In the first case above we may take $\theta_1 = \theta - \varepsilon$ and $\theta_2 = \theta + \varepsilon$. In the second case above we may take $\theta_1 = \pi/2 - \varepsilon$ and $\theta_2 = \pi/2$. Fig. 10 illustrates how θ_1 and θ_2 may be chosen.

THEOREM 2.17. *Any λ with $-\theta_1 \leq \arg\lambda \leq \pi - \theta_2$ is an eigenvalue of P_Γ of multiplicity m if and only if it is a resonance of P of multiplicity m.*

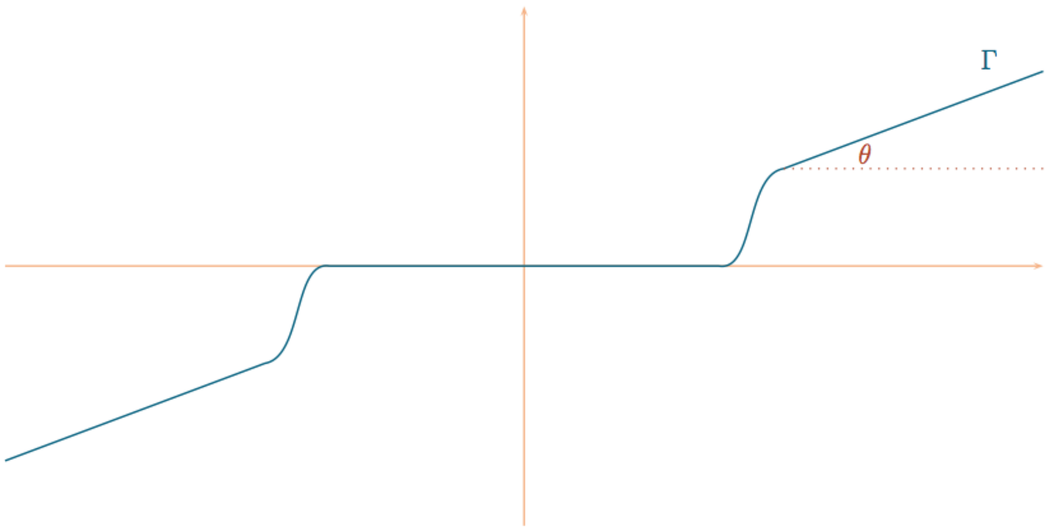

FIGURE 8. Curve Γ used in complex scaling. The curve is given by a C^∞ function g satisfying $g(x) = 0$ for $-R \leq x \leq R$ and $g(x) = x \tan\theta$ for $|x|$ sufficiently large, where θ is a given constant.

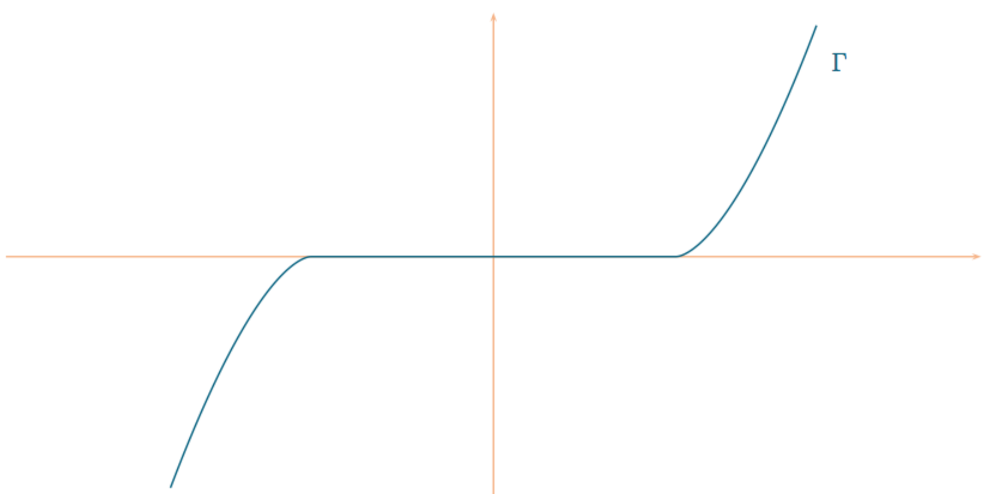

FIGURE 9. Curve Γ used in PML computations. A typical curve is given by a function g satisfying $g(x) = -|x+R|^\alpha$ for $x < -R$, $g(x) = 0$ for $-R \leq x \leq R$, and $g(x) = (x-R)^\alpha$ for $x > R$, where $\alpha > 1$.

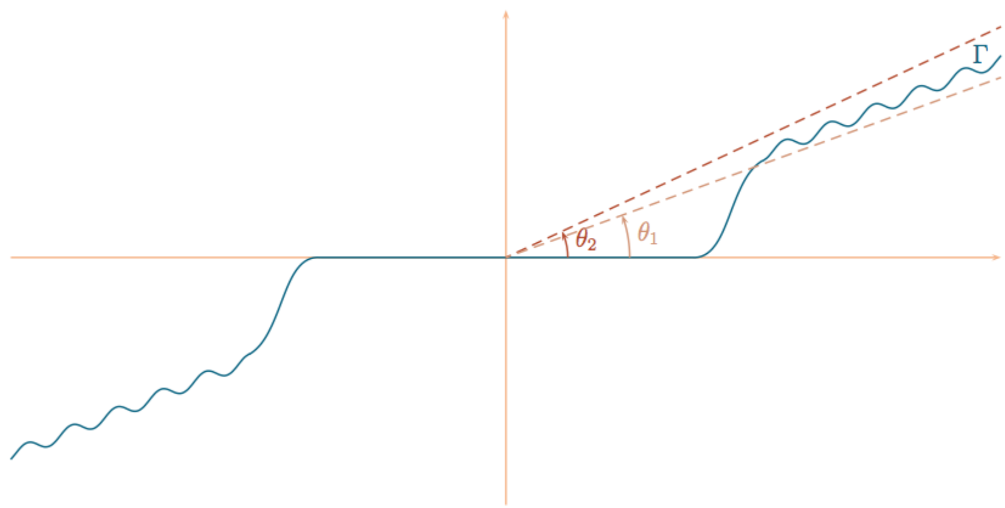

FIGURE 10. A more general version of Γ.

Proof. Suppose λ is a resonance of multiplicity m of P. This means that there is a function $\varphi : \mathbb{R} \to \mathbb{C}$ satisfying $(P - \lambda^2)^m \varphi(x) = 0$ and

$$(P - \lambda^2)^{m-1} \varphi(x) = \begin{cases} Ae^{i\lambda x} & x \geq R \\ Be^{-i\lambda x} & x \leq -R \end{cases}$$

This means that φ satisfies

$$\varphi(x) = \begin{cases} P(x)e^{i\lambda x} & x \geq R \\ Q(x)e^{-i\lambda x} & x \leq -R \end{cases}$$

for suitable polynomials P and Q. We now define $\tilde{\varphi} : \Gamma \to \mathbb{C}$ as follows

$$\tilde{\varphi}(z) = \begin{cases} P(z)e^{i\lambda z} & \mathrm{Re}(z) \geq R \\ \varphi(z) & -R < \mathrm{Re}(z) < R \\ Q(z)e^{-i\lambda z} & \mathrm{Re}(z) \leq -R \end{cases}$$

This $\tilde{\varphi}$ clearly satisfies

$$(P_\Gamma - \lambda^2)^m \tilde{\varphi} = 0 \,, \quad (P_\Gamma - \lambda^2)^{m-1} \tilde{\varphi} \neq 0 \,.$$

If now $\tilde{\varphi} \in L^2(\Gamma)$ this will imply that P_Γ has an eigenvalue of multiplicity at least m at λ. We defer the proof of this lemma momentarily. This means that if λ is a resonance of multiplicity m for P, then it is an eigenvalue of multiplicity at least m for P_Γ. In the same way one shows that if λ is an eigenvalue of multiplicity m for P_Γ, then it is a resonance of multiplicity m for P. Observe that in this one-dimensional problem the only possible multiplicities are algebraic. Geometric multiplicites are ruled out by the rigid form of the eigenfunctions of P_Γ for $|\mathrm{Re}(z)| \geq R$. $\square$

To complete the proof we need only the following lemma.

LEMMA 2.18. *Any continuous function $f : \Gamma \to \mathbb{C}$ satisfying*

$$f(z) = \begin{cases} P(z)e^{i\lambda z} & \mathrm{Re}(z) \geq R \\ Q(z)e^{-i\lambda z} & \mathrm{Re}(z) \leq -R \end{cases}$$

where $P(z)$ and $Q(z)$ are polynomials, is in $L^2(\Gamma)$.

Proof. We must show that

$$\int_\Gamma |f(z)|^2 d|z| < \infty$$

To prove this it is enough to consider the tail of the integral, where $\theta_1 \leq \arg z \leq \theta_2$. Suppose this inequality holds for $\mathrm{Re}(z) \geq T$, and that $T \geq R$, and let

$$\Gamma_T = \Gamma \cap \{\mathrm{Re}(z) \geq T\}\,.$$

Then

$$\begin{aligned}
\int_{\Gamma_T} |f(z)|^2 d|z| &= \int_{\Gamma_T} |P(z)||e^{i\lambda z}|^2 d|z| \\
&= \int_{\Gamma_T} |P(z)|e^{-2(\mathrm{Im}(\lambda)\mathrm{Re}(z)+\mathrm{Re}(\lambda)\mathrm{Im}(z))} d|z| \\
&= \int_{\Gamma_T} |P(z)|e^{-2|z||\lambda|\sin(\arg\lambda+\arg z)} d|z| \\
&\leq \int_{\Gamma_T} |P(z)|e^{-2|z||\lambda|\sin\varepsilon} d|z| \\
&= \int_T^\infty |P(t+ig(t))|e^{-C\sqrt{t^2+g(t)^2}}\sqrt{1+g'(t)^2}dt
\end{aligned}$$

We now reduce to the case $P(z) = 1$. For $|z|$ sufficiently large, $|P(z)| \leq A|z|^n$, and, for a suitable constant C', $A|z|^n e^{-C|z|} \leq e^{-C'z}$ when $|z|$ is sufficiently large. If now $P(z)$ is not 1, we reduce to this case by replacing C by C' and T by T' where T' is large enough that this last inequality holds. We then write

$$[T,\infty) = A \cup B\,,$$
$$A := \{t \geq T : 1 \geq g'(t)\}\,, \quad B := \{t \geq T : 1 \leq g'(t)\}\,.$$

This allows us to divide our integral into two parts which we bound separately:

$$\int_A e^{-C\sqrt{t^2+g(t)^2}}\sqrt{1+g'(t)^2}dt \leq \sqrt{2}\int_A e^{-Ct}dt < \infty$$

On B we use the fact that g is monotone to effect a change of variables:

$$\int_B e^{-C\sqrt{t^2+g(t)^2}}\sqrt{1+g'(t)^2}dt \leq \sqrt{2}\int_T^\infty e^{-Cg(t)}g'(t)dt$$
$$= \sqrt{2}\int_{g(T)}^{g(\infty)} e^{-Cu}du < \infty$$

The proof of convergence of the other side of the integral follows the same line of reasoning. □

2.7. **Sources and further reading.** Theorem 2.11 was proved in some special cases in [Re] and in general (for $V \in L^1_{\rm comp}(\mathbb{R};\mathbb{R})$) in [Z1]. Different proofs were given in [Fr] and [Si], and we followed [Fr], where complex valued potentials were allowed, in our presentation. That paper also treats non-compactly supported potentials.

The presentation of complex scaling in Section 2.6 is based on unpublished notes of Kiril Datchev.

3. **Resonances for potentials in odd dimensions**

In this section we will consider the simplest higher dimensional situation: scattering by compactly supported potentials in odd dimensions. Many results presented in Chapter 2 are valid in this case with proofs requiring only small modifications. Other results, such as the asymptotics for the number of scattering poles, are not known.

The main advantage of odd dimensions is the strong Huyghens principle for the wave equation: if $\Box u = 0$ and the support of initial data lies in $|x| < R$ then support of $u(t, \bullet)$ lies in $t - R < |x| < t + R$. The weak Huyghens principle valid in all dimensions says only that the support of $u(t, \bullet)$ lies in $|x| < t + R$.

One consequence of the strong Hughens principle is the analytic continuation of $(-\Delta - \lambda^2)^{-1}$ from $\mathrm{Im}\lambda > 0$ to $\mathbb{C}$.

3.1. **Free resolvent in odd dimensions.**

We will base our presentation on the properties of the wave equation. Thus we consider its unique forward fundamental solution:

$$\Box E_+ := (\partial_t^2 - \Delta)E_+ = \delta_0(x)\delta_0(t)\,, \quad \mathrm{supp} E_+ \subset \{t \geq 0\}\,. \tag{3.1}$$

For n odd we have a particularly nice expression for the distribution E_+. Its action on $\varphi \in C_c^\infty(\mathbb{R}_t \times \mathbb{R}_x^n)$ is given by

$$\begin{gathered} \langle E_+, \varphi \rangle = \int_0^\infty \langle E(t), \varphi(t, \bullet) \rangle dt\,, \\ \langle E(t), \psi \rangle = \frac{1}{4}\pi^{-k} \left(\frac{d}{ds} \right)^{k-1} \widetilde{\psi}|_{s=t^2}\,, \quad n = 2k+1\,, \\ \widetilde{\psi}(r) := \int_{|\omega|=1} \psi(r\omega) d\omega\,, \end{gathered} \tag{3.2}$$

see [H1, Section 6.2].

The crucial fact seen from this expression is the support property of $E(t)$: for odd $n \geq 3$

$$\mathrm{supp} E(t) = \{(x, t) \;:\; |x| = |t|\}\,. \tag{3.3}$$

This is known as the *strong Huyghens principle*:

$$\begin{gathered} \Box u = f\,,\; \mathrm{supp} f \subset B_{\mathbb{R}^{n+1}}(0, R)\,,\; u|_{t<-R} = 0 \implies \\ u(t, x) = 0\,, \quad \text{for } |x| < t - 2R. \end{gathered}$$

The *weak Huyghens principle* valid in all dimensions says that $u(t, x) = 0$ for $|x| > t + 2R$.

Another way to view $E(t)$ is as giving the solution to the initial value problem:

$$\begin{aligned} &\Box u = 0\,,\ u(0,x) = \varphi_0(x)\,,\quad \partial_t u(0,x) = \varphi_1(x)\,,\\ &u(t,x) = E(t) * \varphi_1(x) + \partial_t E(t) * \varphi_0(x)\,,\quad \varphi_j \in C^\infty(\mathbb{R}^n)\,. \end{aligned} \tag{3.4}$$

Here $u*v$ denotes the convolution of a compactly supported distribution u with a smooth function v.

The solution of (3.4) can also be given using the spectral decomposition of $-\Delta$ and the functional calculus – this corresponds to the Fourier transform decomposition:

$$u(t,x) = \frac{\sin(t\sqrt{-\Delta})}{\sqrt{-\Delta}}\varphi_1(x) + \cos(t\sqrt{-\Delta})\varphi_0(x)\,. \tag{3.5}$$

If we write

$$U(t) := \frac{\sin(t\sqrt{-\Delta})}{\sqrt{-\Delta}}\,,$$

we see that the Schwartz kernel of $U(t)$ is given by

$$U(t,x,y) = E(t,x-y)\,,$$

see [H1, Section 6.1] for the details on the pull back (by $(x,y) \mapsto x-y$ here) of distributions.

The strong Huyghens principle (3.3) implies that

$$\mathrm{supp} v \subset B_{\mathbb{R}^n}(0,R) \implies (U(t)u)(x) = 0\,,\ |x| < t-R\,. \tag{3.6}$$

For future reference we note that the spectral representation immediately gives

$$\partial^k U(t)\ :\ H^s(\mathbb{R}^n) \longrightarrow H^{s-k+1}(\mathbb{R}^n)\,,\quad k \in \mathbb{N}\,,\quad s \in \mathbb{R}\,. \tag{3.7}$$

The outgoing resolvent of the free Laplacian is defined just as in the case of dimension one:

$$R_0(\lambda) := (-\Delta - \lambda^2)^{-1}\ :\ L^2(\mathbb{R}^n) \longrightarrow L^2(\mathbb{R}^n)\,,\quad \mathrm{Im}\lambda > 0\,. \tag{3.8}$$

We can write $R_0(\lambda)$ using $U(t)$:

$$R_0(\lambda) = \int_0^\infty e^{i\lambda t} U(t) dt\,. \tag{3.9}$$

In fact, since $U(t) = \sin t\sqrt{-\Delta}/\sqrt{-\Delta}$, and

$$\sup_{\lambda\in\mathbb{R}} |\sin t\lambda/\lambda| = |t|\,,$$

we have

$$\|U(t)\|_{L^2\to L^2} = \mathcal{O}(|t|)\,.$$

If Im$\lambda > 0$ the right hand side of (3.9) is then bounded on L^2.

This representation gives us the following important result:

THEOREM 3.1 (Free resolvent in odd dimensions). *Suppose that $n \geq 3$ is odd. Then the resolvent defined by*

$$R_0(\lambda) = (-\Delta - \lambda^2)^{-1} \; : \; L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n) \,,$$

for Im$\lambda > 0$*, continues analytically to an entire family of operators*

$$R_0(\lambda) : L^2_{\mathrm{comp}}(\mathbb{R}^n) \longrightarrow L^2_{\mathrm{loc}}(\mathbb{R}^n) \,.$$

For any $\rho \in C^\infty_{\mathrm{c}}(\mathbb{R}^n)$ we have the following estimates:

$$\rho R_0(\lambda)\rho = \mathcal{O}(|\lambda|^{j-1} e^{L \mathrm{Im}\lambda_-}) \; : \; L^2(\mathbb{R}^n) \longrightarrow H^j(\mathbb{R}^n) \,, \tag{3.10}$$

$j = 0, 1, 2$, where $L >$ diam (suppρ)*.*

Proof. 1. For the statement about meromorphy It suffices show that for any $\rho \in C^\infty_{\mathrm{c}}(\mathbb{R}^n)$,

$$\rho R_0(\lambda)\rho : L^2 \longrightarrow L^2$$

continues from Im$\lambda > 0$ to an entire family of bounded operators.

2. If supp$\rho \subset B(0, R)$ then (3.6) and (3.9) show that, for Im$\lambda > 0$ at first,

$$\rho R_0(\lambda)\rho = \int_0^{2R} e^{i\lambda t} \rho U(t) \rho dt \,. \tag{3.11}$$

The right hand side is now defined and holomorphic for $\lambda \in \mathbb{C}$.

3. Since $U(t) = \mathcal{O}_{L^2 \to H^1} = \mathcal{O}(t)$ (see the discussion following (3.9)) we obtain the bound (3.10) for $j = 1$ from (3.11). For $j = 0$ we write

$$\begin{aligned} \lambda \rho R_0(\lambda)\rho &= \int_0^{2R} D_t(e^{i\lambda t}) \rho U(t) \rho dt \\ &= \int_0^{2R} e^{i\lambda t} \rho D_t U(t) \rho dt + i\rho^2 I \,. \end{aligned}$$

We have

$$D_t U(t) = -i \cos t\sqrt{-\Delta} = \mathcal{O}_{L^2 \to L^2}(1) \,,$$

and the bound (3.10) for $j = 0$ follows.

4. Finally, we consider (3.10) for $j = 2$. Suppose that $\rho_1 \in C^\infty_{\mathrm{c}}(\mathbb{R}^n)$ is equal to one on the support of ρ, diam(suppρ_1), where L is the fixed

number appearing in (3.10) and greater than diam (suppρ). Then

$$\begin{aligned}\|\rho R_0(\lambda)\rho\|_{L^2\to H^2} &\leq \|\Delta\rho R_0(\lambda)\rho\|_{L^2\to L^2} + \|\rho R_0(\lambda)\rho\|_{L^2\to L^2} \\ &\leq \|\rho\Delta R_0(\lambda)\rho\|_{L^2\to L^2} + \|[\Delta,\rho](\rho_1 R_0(\lambda)\rho_1)\rho\|_{L^2\to L^2} \\ &\qquad + \|\rho R_0(\lambda)\rho\|_{L^2\to L^2} \\ &\leq |\lambda|^2\|\rho R_0(\lambda)\rho\|_{L^2\to L^2} + C\|\rho_1 R_0(\lambda)\rho_1\|_{L^2\to H^1} \\ &\qquad + 2\|\rho R_0(\lambda)\rho\|_{L^2\to L^2}\,.\end{aligned}$$

Hence (3.10) for $j = 2$ follows from the estimates for $j = 0, 1$. □

The next theorem gives asymptotics of $R_0(\lambda)f$, $f \in \mathcal{E}'(\mathbb{R}^n)$ as $|x| \to \infty$ for $\lambda \neq 0$.

This result does not depend on the parity of the dimension.

THEOREM 3.2 (Outgoing asymptotics). *Suppose that $n > 1$ and $f \in \mathcal{E}'(\mathbb{R}^n)$ is a compactly supported distribution (or $f \in \mathscr{S}(\mathbb{R}^n)$).*

Then for $\lambda \in \mathbb{R}\setminus 0$,

$$R_0(\lambda)f(|x|\theta) = \frac{e^{i\lambda|x|}}{|x|^{\frac{n-1}{2}}}h(|x|,\theta)\,,$$

$$h(x,\theta) \sim \sum_{j=0}^{\infty} |x|^{-j}h_j(\theta)\,, \tag{3.12}$$

$$h_0(\theta) = \frac{1}{2}\frac{1}{2\pi i}\left(\frac{\lambda}{2\pi}\right)^{\frac{1}{2}(n-3)} e^{\frac{1}{4}\pi i(n-1)}\hat{f}(\lambda\theta)\,,$$

as $|x| \to \infty$.

REMARK. In the case of $n = 3$ a simple proof of (3.12) follows from the explicit formula for the Schwartz kernel of $R_0(\lambda)$:

$$R_0(\lambda,x,y) = \frac{e^{i\lambda|x-y|}}{4\pi|x-y|}\,. \tag{3.13}$$

To see this we use the following expansions,

$$|x-y| = |x| - \langle x/|x|, y\rangle + \mathcal{O}\left(1/|x|\right)\,,$$

$$|x-y|^{-1} = |x|^{-1}\left(1+\mathcal{O}\left(|x|^{-1}\right)\right)\,.$$

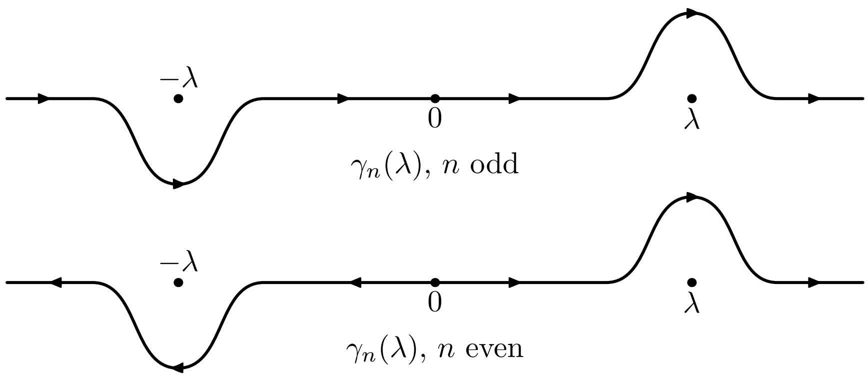

FIGURE 11. Contour deformation used to define $R_0(\lambda)$ for $\lambda > 0$.

Proof. 1. Our proof is based on the representation using the Fourier transform:

$$R_0(\lambda, x, y) = \frac{1}{(2\pi)^n} \int \frac{e^{i\langle \xi, x-y\rangle}}{|\xi|^2 - \lambda^2} d\xi\,, \quad \mathrm{Im}\lambda > 0\,, \tag{3.14}$$

where the integral is meant in the sense of a Fourier transform of a tempered distribution:

$$R_0(\lambda, x, y) = \lim_{\delta \to 0+} \frac{1}{(2\pi)^n} \int \frac{e^{i\langle \xi, x-y\rangle - \delta|\xi|}}{|\xi|^2 - \lambda^2} d\xi\,, \quad \mathrm{Im}\lambda > 0\,, \tag{3.15}$$

2. To obtain an expression valid for $\lambda \in \mathbb{R}_\pm$ we deform the contour in the ξ integration. Let us first assume that $f \in C_c^\infty(\mathbb{R}^n)$ so that

$$|\partial^\alpha f(\xi)| \leq C_{\alpha,N} \langle \xi \rangle^{-N} \exp(R|\mathrm{Im}\xi|)\,, \quad \xi \in \mathbb{C}^n\,.$$

For $\lambda > 0$ and $\epsilon > 0$,

$$\begin{gathered}(2\pi)^n R_0(\lambda + i\epsilon) f(x) = \\ \int_0^\infty \int_{\mathbb{S}^{n-1}} e^{i\rho\langle \omega, x\rangle} (\rho^2 - (\lambda + i\epsilon)^2)^{-1} \hat{f}(\rho\omega) \rho^{n-1} d\omega d\rho = \\ \frac{1}{2} \int_{\mathbb{R}} \int_{\mathbb{S}^{n-1}} e^{i\rho\langle \omega, x\rangle} (\rho^2 - (\lambda + i\epsilon)^2)^{-1} \hat{f}(\rho\omega)\, \mathrm{sgn}(\rho)^{n-1} \rho^{n-1} d\omega d\rho = \\ \frac{1}{2} \lambda^{n-2} \int_{\gamma_n(1)} \int_{\mathbb{S}^{n-1}} e^{i\lambda\rho\langle \omega, x\rangle} (\rho^2 - (1 + i\epsilon)^2)^{-1} \hat{f}(\lambda\rho\omega) \rho^{n-1} d\omega d\rho\,,\end{gathered} \tag{3.16}$$

where the contours for n even and odd are shown in Fig. 11 – note the difference of orientation depending on the parity of dimension. It is clear now that we can take $\epsilon = 0$ which gives an expression for the resolvent on the real axis (see also the remark after the proof).

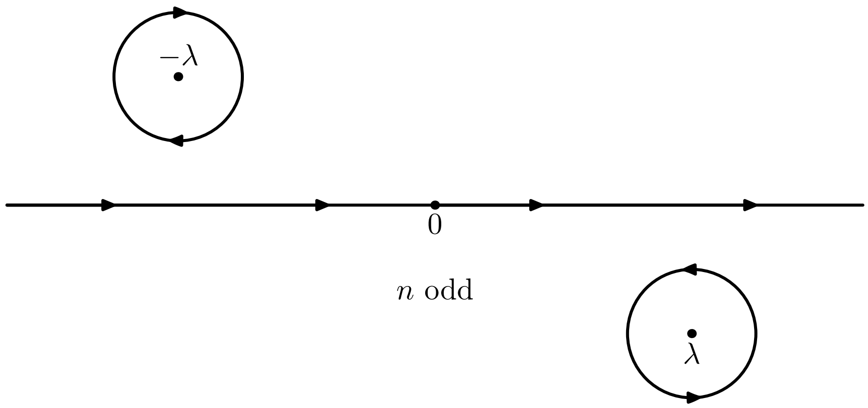

FIGURE 12. Contour deformations used to define $R_0(\lambda)$ for $\lambda > 0$ for n odd. Because of the orientation $R_0(e^{-\pi i}\lambda) = R_0(\lambda)$ for $\lambda > 0$ and the operator is defined in $\mathbb{C}$.

3. We now write $x = r\theta$, $r \geq 0$, $\theta \in \mathbb{S}^{n-1}$, and introduce a partition of unitity on $\mathbb{S}^{n-1}$:

$$\psi_\theta^0(\omega) + \psi_\theta^+(\omega) + \psi_\theta^-(\omega) = 1\,,$$
$$\mathrm{supp}\psi_\theta^\pm \subset \{\omega \;:\; \pm\langle\theta, \omega\rangle > 1/2\}\,.$$

On the support of ψ_θ^0 the phase $\lambda r \rho\langle\theta, \omega$

$\square$

REMARK. The contour deformation given in (3.16) gives also an expression for the analytic continuation of $R_0(\lambda)f$, $f \in C_c^\infty(\mathbb{R}^n)$. The orientation of $\gamma_n(\lambda)$ explains the different behaviour of $R_0(\lambda)f$ for n even and odd.

3.2. **Meromorphic continuation.**

Once we have established the properties of the free resolvent in odd dimensions the properties of

$$R_V(\lambda) := (-\Delta + V - \lambda^2)^{-1}\,, \quad \mathrm{Im}\lambda \gg 0\,,$$
$$V \in L^\infty(\mathbb{R}^n, \mathbb{C})\,, \quad n = 2k+1\,,\; k = 1, 2, \cdots\,,$$

follow exactly as in one dimension. The situation is even simpler as we do not have a resonance at zero for $R_0(\lambda)$.

In particular the proof of the following theorem is exactly the same as in one dimensional case:

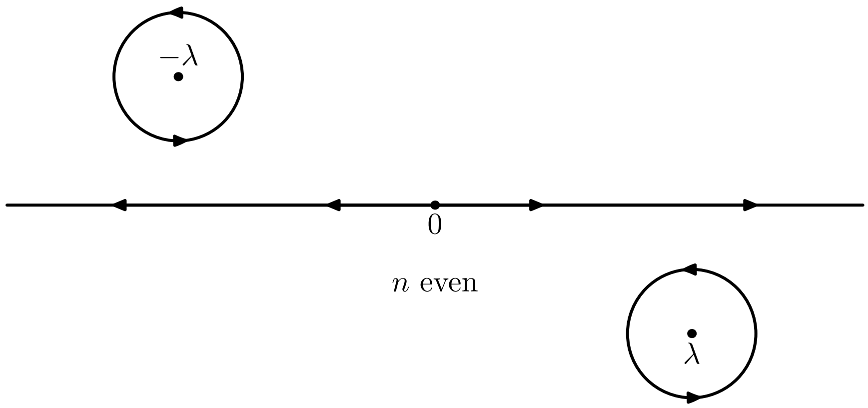

FIGURE 13. Contour deformations used to define $R_0(\lambda)$ for $\lambda > 0$ for n even. Now $R_0(e^{-\pi i}\lambda) = R_0(\lambda)$ can be expressed using an integral ovegral over the circular contour which doubles rather than gets absorbed. The resolvent is defined on the logarithmic plane.

THEOREM 3.3 (Meromorphic continuation of the resolvent II). *Suppose that $V \in L^\infty_{\rm comp}(\mathbb{R}^n; \mathbb{C})$ and that $n \geq 3$ is odd. Then the*

$$R_V := (D_x^2 + V - \lambda^2)^{-1} \; : \; L^2 \longrightarrow L^2 \,, \quad {\rm Im}\lambda > 0 \,,$$

is a meromorphic family of operators with singularities contained in $D(0, R_V)$ for some V.

It extends to a meromorphic family of operators for $\lambda \in \mathbb{C}$:

$$R_V := L^2_{\rm comp} \longrightarrow L^2_{\rm loc} \,.$$

Scattering resonances are the poles of $R_V(\lambda$ and their multiplicities, $m_R(\lambda)$ are defined by (2.13). The structure of the singular part of the resolvent at a pole can now be more complicated:

THEOREM 3.4 (Singular part of $\boldsymbol{R_V(\lambda)}$ II).
1) Suppose $m_R(\mu) > 0$, $\mu \neq 0$. Then

$$R_V(\lambda) = \sum_{k=1}^{m_R(\mu)} \frac{(P - \mu^2)^{k-1}}{(\lambda^2 - \mu^2)^k} \Pi_\mu + A(\lambda, \mu) \,, \tag{3.17}$$

where $\lambda \mapsto A(\lambda, \mu)$ is holomorphic near μ,

$$\Pi_\mu = \frac{1}{2\pi i} \oint_\mu R_V(\lambda) 2\lambda d\lambda \,,$$

and

$$(3.18)\qquad (P_V-\mu^2)^{m_R(\mu)}\Pi_\mu=0\,,\quad \mathrm{Im}\Pi_\mu=\mathrm{span}\,\{u_1,\cdots,u_{m_R(\mu)}\}\,.$$

The non-empty range of $(P_V-\mu^2)^{m_R(\mu)-1}\Pi_\mu$ *consists of outgoing solutions to* $(P_V-\mu^2)u=0$.

2) Suppose that $V\in L^\infty_{\mathrm{comp}}(\mathbb{R};\mathbb{R})$ *and that* $m_R(0)>0$. *Then*

$$R_V(\lambda)=\frac{\Pi_0}{\lambda^2}+\frac{\Pi_0^R}{\lambda}+A(\lambda)\,,$$

where $\lambda\mapsto A(\lambda)$ *is holomorphic near* 0, *and* Π_0 *is the orthogonal projection onto the space of* L^2 *solutions to* $P_Vu=0$. *The range of* Π_0^R *consists of outgoing solutions to* $P_Vu=0$ *which are* not *in* L^2.

3) For $n\geq 5$, *all outgoing solutions at* $\mu=0$ *are in* L^2. *In other words a pole of* $R_V(\lambda)$ *at* $\lambda=0$ *comes the existence of a zero eigenvalue..*

Proof. 1. The proof of 1) is the same as the proof of 1) in Theorem 2.3.

2. As in the proof of Theorem 2.3 we see that

$$R_V(\lambda)=\frac{A_2}{\lambda^2}+\frac{A_1}{\lambda}+A(\lambda)\,,$$

where $P_VA_j=A_jP_V=0$, $P_VA(0)=I+A_2$ and $A(\lambda)$ is holomorphic near 0. We also have

$$A_2=-\lim_{t\to 0+}t^2R_V(it)\,,$$

which shows that A_2 is bounded on L^2 and selfadjoint. **more needed here**

3. The outgoing solutions are of the form $u=R_0(0)f$ where $f\in L^2_{\mathrm{comp}}$. Hence

$$u(x)\sim\frac{c_n}{|x|^{n-2}}\int f(y)dy\,,\quad x\to\infty\,,$$

more details needed here and about $R_0(\lambda)$. Hence for $n\geq 5$, $u\in L^2$. □

The proofs of Theorem 2.6 on resonance free regions and of Theorem 2.5 apply without any modifications to the case of higher odd dimensions. Thus we obtain

THEOREM 3.5 (Resonance free regions II). *Suppose that*

$$V \in L^\infty_{\rm comp}(\mathbb{R}^n;\mathbb{C})\,, \quad n \geq 1\,, \quad odd\,.$$

Then for any $\rho \in C^\infty_{\rm c}(\mathbb{R}^3)$ constants A', C, T depending on the support of ρ such that

$$\|\rho R_V(\lambda)\rho\|_{L^2\to H^j} \leq C|\lambda|^{j-1}\, e^{T|\operatorname{Im}\lambda|}\,, \quad j=0,1,2\,, \tag{3.19}$$

for

$$\operatorname{Im}\lambda \geq -A - \delta\log(1+|\lambda|)\,, \quad |\lambda| > C_0\,, \quad \delta > 1/|\operatorname{chsupp} V|\,.$$

In particular there are only finitely many resonances in the region

$$\{\lambda \;:\; \operatorname{Im}\lambda \geq -A - \delta\log(1+|\lambda|)\}\,.$$

for any $A > 0$.

THEOREM 3.6 (Resonance expansions of scattering waves II). *Let $V \in L^\infty(\mathbb{R}^n;\mathbb{R})$ for $n \geq 1$ odd, and suppose that $w(t,x)$ is the solution of*

$$\left\{\begin{array}{l} (D_t^2 - P_V)w(t,x) = 0\,, \\ w(0,x) = w_0(x) \in H^1_{\rm comp}(\mathbb{R})\,, \\ \partial_t w(0,x) = w_1(x) \in L^2_{\rm comp}(\mathbb{R})\,. \end{array}\right. \tag{3.20}$$

Let $E_N < \cdots < E_1 < 0$ be the negative eigenvalues of P_V and $\{\lambda_j\} \subset \{\operatorname{Im}\lambda < 0\}$ be the set of its resonances.

Then, for any $A > 0$,

$$\begin{aligned} w(t,x) = {} & \sum_{k=1}^N \cosh(t\sqrt{-E_k})a_k v_k(x) + \sum_{k=1}^N \sinh(t\sqrt{-E_k})b_k v_k(x) \\ & + \sum_{\operatorname{Im}\lambda_j > -A} \; \sum_{\ell=0}^{m_R(\lambda_j)-1} \lambda_j^\ell e^{-i\lambda_j t} w_{j,\ell}(x) + E_A(t)\,, \end{aligned} \tag{3.21}$$

where the second sum is finite,

$$\sum_{\ell=0}^{m_R(\lambda_j)-1} \lambda_j^\ell e^{-i\lambda_j t} w_{j,\ell}(x) = \operatorname{Res}_{\lambda=\lambda_j}\left((iR_V(\lambda)w_1 + \lambda R_V(\lambda)w_0)\, e^{-i\lambda t}\right)\,, \tag{3.22}$$

$$(P_V - \lambda_j)^{k+1} w_{j,k} = 0\,,$$

and for any $K > 0$*, such that* $\text{supp} w_j \subset [-K, K]$*, there exist constants* $C_{K,A}$ *and* $T_{K,A}$

$$\|E_A(t)\|_{H^2([-K,K])} \leq C_{K,A} e^{-tA} \left(\|w_0\|_{H^1} + \|w_1\|_{L^2}\right) , \quad t \geq T_{K,A} .$$

3.3. **Upper bounds on the number of resonances.**

As in the case of dimension one we will estimate the number of resonances using a suitable determinant. We start with

LEMMA 3.7 (**Trace class properties**). *For* $V, \rho \in L^\infty_{\text{comp}}(\mathbb{R}^n; \mathbb{C})$, $n \geq 1$*, odd,*

$$(V R_0(\lambda) \rho)^p , \quad p \geq \frac{n+1}{2} ,$$

is an entire family of trace class operators.

Proof. 1. We first estimate the characteristic values of $\rho_1 R_0(\lambda) \rho_1$ where $\rho_1 \in C^\infty_{\text{c}}(\mathbb{R}^n)$. If $\text{supp}\rho_1 \subset B(0, R)$ we can consider

$$\rho_1 R_0(\lambda) \rho_1 : L^2(\mathbb{T}^n_R) \longrightarrow L^2(\mathbb{T}^n_R) , \quad \mathbb{T}_R := \mathbb{R}^n / R\mathbb{Z}^n . \tag{3.23}$$

2. Then, using (B.13), and then (B.16), we have

$$\begin{aligned} s_j(\rho_1 R_0(\lambda) \rho_1) &\leq s_j((-\Delta_{\mathbb{T}^n_R} - 1)^{-\ell}) \|(-\Delta_{\mathbb{T}^n_R} - 1)^\ell \rho_1 R_0(\lambda) \rho_1\| \\ &\leq C j^{-2\ell/n} \|\rho_1 R_0(\lambda) \rho_1\|_{L^2 \to H^{2\ell}} . \end{aligned}$$

Theorem 3.1 gives

$$s_j(\rho_1 R_0(\lambda) \rho_1) \leq C \min(|\lambda|^{-1}, j^{-1/n}, |\lambda| j^{-2/n}) \exp(C \text{Im} \lambda_-) . \tag{3.24}$$

3. The same estimate holds for $V R_0(\lambda) \rho$ and we can use(B.13) to see that

$$s_j \left((V R_0(\lambda) \rho)^p \right) \leq C_1 |\lambda|^p j^{-2p/n} \exp(C_1 \text{Im} \lambda_-) .$$

when $p \geq (n+1)/2$

$$\sum_j s_j \left((V R_0(\lambda) \rho)^p \right) < \infty$$

which means the operator is of trace class. $\square$

Since for $n \geq 2$, $VR_0(\lambda)$ is no longer of trace class we cannot use the determinant defined by (2.20).

DEFINITION. Suppose that $n \geq 3$ is odd. Using the modified Fredhold determinant, see (B.21) in Section B.3, Lemma 3.7 allows us to define

$$D(\lambda) := \det_p(I + VR_0(\lambda)\rho)\,, \quad p = \frac{n+1}{2}\,, \tag{3.25}$$

where $\rho \in C^\infty_{\mathrm{c}}(\mathbb{R}^n)$ is equal to one on $\mathrm{supp}V$.

Using Lemma 3.7 the following definition is also justified:

$$H(\lambda) := \det(I - (VR_0(\lambda)\rho)^{n+1})\,. \tag{3.26}$$

THEOREM 3.8 (Multiplicity of a resonance II). *Let the functions D and H be given by* (3.25) *and* (3.26) *respectively.*

Let $m_D(\lambda)$ and $m_H(\lambda)$ be multiplicities of λ as zeros of $D(\lambda)$ and $H(\lambda)$, respectively.

Then for $\lambda \in \mathbb{C}$,

$$m_R(\lambda) = m_D(\lambda) \leq m_H(\lambda)\,. \tag{3.27}$$

Proof. 1. Since

$$\begin{aligned} &R_V(\lambda) = R_0(\lambda)(I + VR_0(\lambda)\rho)^{-1}(I - VR_0(\lambda)(1-\rho))\,, \\ &(I + VR_0(\lambda)\rho)^{-1} = I - VR_V(\lambda)\rho\,. \end{aligned} \tag{3.28}$$

the poles of R_V and $(I + VR_0(\lambda)\rho)^{-1}$ coincide.

Also, as n is odd, $I - (VR_0(\lambda)\rho)^{n+1} =$

$$(I + VR_0(\lambda)\rho)(I - VR_0(\lambda)\rho + \cdots - (VR_0(\lambda)\rho)^n)\,.$$

2. The study of multiplicities needs to be based on a more careful argument using the results of Section C.4. □

DISCUSSION. In view of (3.27) the advantage of $D(\lambda)$ is that it gives us resonances with their multiplicities. As we will see in Section 3.6 $D(\lambda)$ grows too fast as $\mathrm{Im}\lambda \to -\infty$ (except when $n = 3$). This makes estimates on the number of its zeros unyieldy.

The determinant $H(\lambda)$ is introduced to remedy the growth problem but we pay by introducing additional zeros. For bounds on the growth of the number of resonances, which is all we are able to do precisely, that of course does not matter. The choice of $n+1$ as the power of $VR_0(\lambda)\rho$ was arbitrary as in view of Lemma 3.7 we could have taken any $p \geq (n+1)/2$. It turns out convenient in the proof of Theorem 3.10.

The main result of this section is the following upper bound

THEOREM 3.9 (**Upper bounds on the number of resonances I**). *Suppose that $n \geq 3$ is odd and that $V \in L^\infty_{\rm comp}(\mathbb{R}^n;\mathbb{C})$. Let $m_R(\lambda)$ be the multiplicity of a resonance at λ as defined in* (2.13).

Then

$$\sum \{ m_R(\lambda) \; : \; |\lambda| \leq r \} \leq C_V r^n \,. \tag{3.29}$$

INTERPRETATION. In the case of $-\Delta + V$ on a bounded domain, for instance on $\mathbb{T}^n$, the spectrum is discrete and for $V \in L^\infty(\mathbb{T}^n;\mathbb{R})$ we have the asymptotic Weyl law for the number of eigenvalues:

$$\begin{gathered} |\{ \lambda : \lambda^2 \in \operatorname{Spec}(-\Delta_{\mathbb{T}^n} + V) \,, \; |\lambda| \leq r \}| = c_n {\rm vol}\,(\mathbb{T}^n) r^n (1 + o(1)) \,, \\ c_n = 2{\rm vol}\,(B_{\mathbb{R}^n}(0,1))/(2\pi)^n \,, \end{gathered}$$

where the eigenvalues are included according to their multiplicities.

In the case of $-\Delta + V$ on $\mathbb{R}^n$ the discrete spectrum is replaced by the discrete set of resonances. Hence the bound (3.29) is an analogue of the Weyl law. Except in dimension one (see Theorem 2.11) the issue of asymptotics or even optimal lower bounds remains unclear at the time of writing (see Section 3.8 for references).

Jensen's formula, see (D.1) in Section D, and (3.27) show that Theorem 3.9 is an immediate consequence of an estimate on $H(\lambda)$:

THEOREM 3.10 (**Determinant bounds I**). *Let $H(\lambda)$ be given by* (3.26). *Then for some constant $A = A_V$,*

$$|H(\lambda)| \leq A \exp(A |\lambda|^n) \,. \tag{3.30}$$

Proof. 1. We use the Weyl inequality (B.14) to see that

$$|H(\lambda)| \leq \prod_{k=1}^\infty \left(1 + s_k((VR_0(\lambda)\rho)^{n+1}) \right) \,. \tag{3.31}$$

We then use (B.13) to see that

$$s_k((VR_0(\lambda)\rho)^{n+1}) \leq \|V\|_\infty^{n+1}\left(s_{[k/(n+1)]}(\rho R_0(\lambda)\rho)\right)^{n+1} . \tag{3.32}$$

Hence we need to estimate $s_j(\rho R_0(\lambda)\rho)$ for $\rho \in C^\infty_{\rm c}(\mathbb{R}^n)$.

2. We start with easier estimates in the physical half-plane $\mathrm{Im}\lambda \geq 0$. We apply (3.24) to obtain

$$s_j(\rho R_0(\lambda)\rho) \leq Cj^{-1/n} .$$

From (3.32) we obtain

$$s_k((VR_0(\lambda)\rho)^{n+1}) \leq C_1 k^{-(n+1)/n} .$$

Using this in (3.31) we then get

$$\begin{aligned} H(\lambda) &\leq \exp\left(\sum_{k=1}^\infty s_k((VR_0(\lambda)\rho)^{n+1})\right) \\ &\leq \exp\left(C_1\sum_{k=1}^\infty k^{-(n+1)/n}\right) \\ &\leq C_2 , \end{aligned}$$

that is $H(\lambda)$ is uniformly bounded for $\mathrm{Im}\lambda \geq 0$.

3. To obtain estimates for $\mathrm{Im}\lambda < 0$ we use the formula

$$\begin{gathered} \rho(R_0(\lambda) - R_0(-\lambda))\rho = a_n\lambda^{n-1}E_\rho(\bar\lambda)^* E_\rho(\lambda) , \\ E_\rho(\lambda)u(\omega) := \int_{\mathbb{R}^n} e^{i\lambda\langle\omega,x\rangle}\rho(x)u(x)dx , \\ E_\rho(\lambda) : L^2(\mathbb{R}^n) \longrightarrow L^2(\mathbb{S}^{n-1}) . \end{gathered} \tag{3.33}$$

Hence for $\mathrm{Im}\lambda < 0$ we have

$$\begin{aligned} s_j(\rho R_0(\lambda)\rho) &\leq a_n|\lambda|^{n-2}\|E_\rho(\lambda)\| s_{[j/2]}(E_\rho(\lambda)) + s_{[j/2]}(\rho R_0(-\lambda)\rho) \\ &\leq C\exp(C|\lambda|)s_{[j/2]}(E_\rho(\lambda)) + Cj^{-1/n} . \end{aligned} \tag{3.34}$$

4. To estimate $s_j(E_\rho(\lambda))$ we use the Laplacian on the sphere, $-\Delta_{\mathbb{S}^{n-1}}$, and (B.13):

$$\begin{aligned} s_j(E_\rho(\lambda)) &\leq s_j((-\Delta_{\mathbb{S}^{n-1}} - 1)^{-\ell})\|(-\Delta_{\mathbb{S}^{n-1}} - 1)^\ell E_\rho(\lambda)\| \\ &\leq C^\ell j^{-2\ell/(n-1)}\|(-\Delta_{\mathbb{S}^{n-1}} - 1)^\ell E_\rho(\lambda)\| \\ &\leq C_1^\ell j^{-2\ell/(n-1)}\exp(C_1|\lambda|)(2\ell)! . \end{aligned} \tag{3.35}$$

He re we used the fact that for ρ with support in $B(0,R)$,

$$\|(-\Delta_{\mathbb{S}^{n-1}}-1)^{\ell}E_\rho(\lambda)\| \leq C_\rho \sup_{\omega\in\mathbb{S}^{n-1},|x|\leq R} \left|(-\Delta_\omega-1)^{\ell}e^{i\lambda\langle x,\omega\rangle}\right| ,$$

and we estimated sup using, essentially, the Cauchy estimates.

We now optimize the estimate (3.35) in ℓ. This gives

$$s_j(E_\rho(\lambda)) \leq C_2 \exp\left(C_2|\lambda| - j^{\frac{1}{n-1}}/C_2\right) . \tag{3.36}$$

5. Going back to (3.32) and (3.34) we obtain

$$s_k((VR_0(\lambda)\rho)^{n+1}) \leq C_3 \exp\left(C_3|\lambda| - k^{\frac{1}{n-1}}/C_3\right) + C_3 k^{-\frac{n+1}{n}} .$$

In particular,

$$s_k((VR_0(\lambda)\rho)^{n+1}) \leq \begin{cases} C_4\exp(C_4|\lambda|)\,, & k\leq C_4|\lambda|^{n-1} \\ C_4k^{-\frac{n+1}{n}}\,, & k\geq C_4|\lambda|^{n-1}\,. \end{cases} \tag{3.37}$$

Returning to (3.31) we use (3.37) as follows

$$\begin{aligned} |H(\lambda)| &\leq \prod_{k\leq C_k|\lambda|^{n-1}} \exp(C_4|\lambda|) \left(\exp \sum_{k\geq C_4|\lambda|^{n-1}} C_4 k^{-(n+1)/n}\right) \\ &\leq \exp(C_5|\lambda|^n)\,, \end{aligned}$$

which completes the proof. □

REMARK. The exponent n in (3.29) is optimal as shown by the case of radial potentials. When $V(x) = v(|x|)(R-|x|)^0_+$, where v is a C^2 even function, and $v(R)>0$. Then, see [Z2],

$$\sum\{m_R(\lambda)\ :\ |\lambda|\leq r\} = C_R r^n(1+o(1))\,. \tag{3.38}$$

The constant C_R and its appearance in (3.29) is explained and discussed in [Ste].

3.4. **Complex valued potentials with no resonances.**

As we have seen in Theorem 2.11 one dimensional complex valued compactly supported non-zero potentials always have infinitely many resonances satisfying nice asymptotics at infinity. The situation is dramatically different in higher dimensions where complex valued potentials may have *no* resonances at all.

THEOREM 3.11 (Complex valued potentials with no resonances). *Let (r, θ, x') be cylindrical coordinates in $\mathbb{R}^{k+2}$, where k is odd:*

$$x = (x_1, x_2, x')\,, \quad x_1 = r\cos\theta\,, \quad x_2 = r\sin\theta\,, \quad x' \in \mathbb{R}^k\,.$$

Suppose that $V \in L^\infty_{\rm comp}(\mathbb{R}^3; \mathbb{C})$ is of the following form:

$$V(x) = e^{i\theta m} W(r, x')\,, \quad W \in L^\infty_{\rm comp}([0,\infty) \times \mathbb{R}^k)\,.$$

If $m \neq 0$ then the resolvent $R_V(\lambda)$ is entire in $\mathbb{C}$, that is the operator $-\Delta + V$ has no resonances.

REMARK. We can easily place conditions on W so that

$$V \in C^\infty_{\rm c}(\mathbb{R}^{2+k}; \mathbb{C})\,.$$

Before starting the proof we need two simple lemmas

LEMMA 3.12 (Fourier decomposition of the resolvent). *Let Π_ℓ be the projection onto the ℓ'th Fourier mode:*

$$\Pi_\ell u(r, \theta, x') := e^{i\ell\theta} \frac{1}{2\pi} \int_0^{2\pi} u(r, \varphi, x') e^{-i\ell\varphi} d\varphi\,. \tag{3.39}$$

Then for $\rho \in C^\infty_{\rm c}(\mathbb{R}^{2+k})$, $\rho = \rho(r, x')$, we have

$$\|\Pi_\ell \rho R_0(\lambda) \rho \Pi_\ell\|_{L^2 \to L^2} \leq \frac{C|\lambda| e^{C \operatorname{Im}\lambda_-}}{1 + \ell^2}\,, \quad \ell \in \mathbb{Z}\,. \tag{3.40}$$

Proof. 1. Because we chose ρ to be independent of θ, Π_ℓ commutes with $\rho R_0(\lambda)\rho$. Put

$$u := \rho R_0(\lambda) \rho \Pi_\ell f\,, \quad f \in L^2\,.$$

Then (3.10) gives

$$\|u\|_{H^2} \leq C|\lambda| e^{C \operatorname{Im}\lambda_-} \|f\|_{L^2}\,. \tag{3.41}$$

2. On the other hand

$$\begin{aligned}\|u\|_{H^2} &\geq \langle -\Delta u, u\rangle = \langle D_r^2 - (i/r)D_r - \Delta_{x'} + \ell^2/r^2)u, u\rangle_{L^2}\\ &\geq \langle(\ell^2/r^2)u, u\rangle_{L^2} \geq C\ell^2\|u\|_{L^2}\,,\end{aligned}$$

where the last inequality followed from the fact that r is bounded on the support of u and hence $\ell^2/r^2 \geq C\ell^2$. Combining this with (3.41) proves (3.40). □

The next lemma is an elementary statement about sequences:

LEMMA 3.13 (Two sided sequences). *Let $\{a_j\}_{j=-\infty}^{\infty}$ be a sequence satisfying*

$$a_j \longrightarrow 0\,, \quad j \longrightarrow \pm\infty\,.$$

Suppose that $m \in \mathbb{Z} \setminus \{0\}$ and that for each j there exists $C_j \geq 0$ such that

$$|a_{j+m}| \leq C_j|a_j|\,, \quad \text{and } C_j \leq 1 \text{ for } |j| \geq J, \tag{3.42}$$

for some J.

Then

$$a_j = 0 \ \text{ for all } j \in \mathbb{Z}.$$

Proof. Fix $j \in \mathbb{Z}$ and use (3.42) to obtain

$$\begin{aligned}|a_j| &\leq C_{j-m}|a_{j-m}| \leq \cdots \leq \prod_{k=1}^{p} C_{j-km}|a_{j-mp}|\\ &\leq K_j|a_{j-mp}| \to 0\,, \quad p \to \infty\,, \ K_j := \prod_{|j-km|<J} C_{j-km}\,.\end{aligned}$$

This shows that $a_j = 0$ as claimed. □

Proof of Theorem 3.11. 1. In view of (3.28) of $m_V(\lambda) > 0$ for for some λ then $(I + VR_0\rho)^{-1}$ has a pole for any $\rho \in C_{\rm c}^{\infty}(\mathbb{R}^{2+k}$ such that $\rho = 1$ on supp$V$. In particular we can take $\rho = \rho(r, x')$.

Hence there exists $u \in L^2$ such that

$$u = -VR_0(\lambda)\rho u = -V\rho R_0(\lambda)\rho u\,.$$

2. We now use the structure of V, $V(r,\theta,x') = e^{im\theta}W(r,x')$, to calculate

$$\begin{aligned}\Pi_{j+m}u &= \Pi_{j+m}\left(e^{im\theta}W\rho R_0(\lambda)\rho u\right)\\ &= e^{im\theta}\Pi_j W\rho R_0(\lambda)\rho\left(\Pi_j u\right)\,.\end{aligned}$$

Lemma 3.12 now shows that

$$\|\Pi_{j+m}u\|_{L^2} \leq \frac{C|\lambda|e^{C|\lambda|}}{1+j^2}\|\Pi_j u\|_{L^2}\,.$$

If we put

$$a_j := \|\Pi_j u\|_{L^2}\,, \quad C_j := \frac{C|\lambda|e^{C|\lambda|}}{1+j^2}\,,$$

then the assumptions of Lemma 3.13 are satisfied. Thus $\Pi_j u = 0$ for all j which means that $u = 0$ and there is no resonance at λ. □

3.5. **Scattering matrix in potential scattering.**

In this section we will define and describe the scattering matrix for $V \in L^\infty_{\rm comp}(\mathbb{R}^n;\mathbb{R})$, $n \geq 3$, odd. Except for the behaviour near $\lambda = 0$ and the fact that we use the properties of the resolvent, the parity of the dimension is not very important here.

In Section 2.3 the scattering matrix mapped incoming to outgoing components of a solution to the generalized eigenvalue equation

$$(P_V - \lambda^2)w = 0\,. \tag{3.43}$$

A conceptually similar procedure is used in the case of scattering in higher dimensions with asymptotic formulae such as (3.12) replacing explicit representations in terms of $\exp(\pm i\lambda x)$. The starting point is the same as in (2.37): we consider solutions to (3.43) of the form

$$w(x,\lambda,\omega) = e^{i\lambda\langle x,\omega\rangle} + u(x,\lambda,\omega)\,, \tag{3.44}$$

where u is outgoing. It is obtained using the resolvent $R_V(\lambda)$, except at the possible poles:

$$u(x,\lambda,\omega) := -R_V(\lambda)(Ve^{i\lambda\langle\bullet,\omega\rangle})\,. \tag{3.45}$$

The condition of being outgoing can be formulated in the following equivalent ways

THEOREM 3.14 (**Outgoing solutions).** *Suppose that $f \in \mathcal{E}'(\mathbb{R}^n)$ is a compactly supported distribution and suppose that u solves*

$$(P_V - \lambda^2)u = f\,, \quad \lambda \in \mathbb{R}\setminus\{0\}\,. \tag{3.46}$$

Then the following conditions are equivalent:

i) $u = R_0(\lambda)g$ for some $g \in \mathcal{E}'(\mathbb{R}^n)$,

ii) $u(x) = e^{i\lambda|x|}a(x/|x|)|x|^{-(n-1)/2} + \mathcal{O}(|x|^{-(n+1)/2})$, as $|x| \to \infty$, *iii)* $(\partial/\partial r - i\lambda)u = o(r^{-(n-1)/2})$, *as* $r \to \infty$, $r = |x|$,

iv) $u = U(\lambda)$ *where* $(P_V - \mu^2)U(\mu) = f$ *and* $U(\mu)$ *is meromorphic in* $\mathrm{Im}\mu \geq 0$, $U(\mu) \in L^2(\mathbb{R}^n)$ *for* $\mathrm{Im}\mu > 0$.

v) $u = R_V(\lambda)f$.

INTERPRETATION. The expression in part ii) of the theorem is interpreted as an outgoing *spherical wave* with $a(x/|x|)$ giving the intensity at different directions $x/|x|$.

DEFINITION. A solution to (3.46) satisfying the conditions in theorem 3.14 is called *outgoing*.

In particular we see that u given by (3.45) is outgoing provided that λ is not a pole of $R_V(\lambda)$:

$$u(x, \lambda, \omega) = -R_V(\lambda)(Ve^{i\lambda\langle \bullet, \omega\rangle}) = R_0(\lambda)f \,,$$
$$f = -(I + VR_0(\lambda)\rho)^{-1}(Ve^{i\lambda\langle \bullet, \omega\rangle}) \in \mathcal{E}'(\mathbb{R}^n) \,.$$

When V is real valued and $\lambda \in \mathbb{R}\backslash\{0\}$ then Rellich's important result states that there are no outgoing solutions. In other words, $R_V(\lambda)$ has no non-zero real poles:

THEOREM 3.15 (Rellich's uniqueness theorem). *Suppose that* $V \in L^\infty_{\mathrm{comp}}(\mathbb{R}^n; \mathbb{R})$ *is real valued. Then for* $\lambda \in \mathbb{R} \setminus \{0\}$ *there are* no *outgoing solutions to*

$$(P_V - \lambda^2)u = 0 \,.$$

Equivalently, $R_V(\lambda)$ *has no poles for* $\lambda \in \mathbb{R} \setminus \{0\}$.

Proof. 1. The proof proceeds by contradiction. Having an outgoing solution to (3.43) means that $R_V(\lambda)$ has a pole at λ **need to get a proper reference earlier in the text**, and that in turn implies that there exists $w \in L^2$ for which

$$(I + \rho R_0(\lambda)V)w = 0 \,,$$

and this is true which is equal to 1 on the support of V. Hence w is defined in $L^2_{\mathrm{loc}}xsy(\mathbb{R}^n)$, $w|_{\mathbb{R}^n \setminus B(0,R)} \in C^\infty$,

$$(P_V - \lambda^2)w = 0 \,, \quad w = R_0(\lambda)Vw \,.$$

more discussion needed here

2. Theorem 3.2 shows that

$$w = R_0(\lambda)(Vw)(x) = \frac{e^{i\lambda|x|}}{|x|^{\frac{n-1}{2}}}\left(h\left(\frac{x}{|x|}\right) + \mathcal{O}\left(\frac{1}{|x|}\right)\right), \tag{3.47}$$

where

$$h(\theta) = c_n \lambda^{\frac{n-3}{2}} \widehat{Vw}(\lambda\theta) .$$

In particular,

$$(\partial_r - i\lambda)w = \mathcal{O}(r^{-\frac{n+1}{2}}) . \tag{3.48}$$

3. Since λ is real we have

$$\begin{aligned} 0 &= \int_{B(0,R)} (\bar{u}(P_V - \lambda^2)u - (P_V - \lambda^2)\bar{u}u)dx \\ &= \int_{B(0,R)} (u\Delta\bar{u} - \bar{u}\Delta u)dx = \int_{\partial B(0,R)} (\partial_r u\bar{u} - u\partial_r\bar{u})dS \end{aligned}$$

Using (3.47) and (3.48) we obtain

$$2i\lambda \int_{\partial B(0,R)} |u|^2 dS = \mathcal{O}(R^{-n}\mathrm{vol}\,(\partial B(0,R))) = \mathcal{O}(R^{-1}) ,$$

which implies (in the notation of (3.47)) that

$$0 = \int_{\mathbb{S}^{n-1}} |h(\theta)|^2 d\theta = |c_n|^2 |\lambda|^{n-3} \int_{\mathbb{S}^{n-1}} |\widehat{Vw}(\lambda\theta)|^2 d\theta .$$

4. We conclude that

$$\widehat{Vw}(\xi) = 0 , \quad \langle \xi, \xi \rangle = \lambda^2 , \quad \xi \in \mathbb{R}^n .$$

If we put

$$\Sigma := \{\xi \in \mathbb{C}^n \ : \ \langle \xi, \xi \rangle = \lambda^2\} ,$$

then Σ is a connected complex hypersurface in $\mathbb{C}^n$ and the entire function $\widehat{Vw}(\xi)$ vanishes on $\Sigma \cap \mathbb{R}^n$. It follows that $\widehat{Vw}(\xi) = 0$ on Σ. From that we see that

$$\frac{\widehat{Vw}(\xi)}{\langle \xi, \xi \rangle - \lambda^2} \ \text{ is an entire function of } \xi \in \mathbb{C}^n.$$

Since

$$(\langle \xi, \xi \rangle - \lambda^2)\widehat{w}(\xi) = \widehat{Vw}(\xi) ,$$

Paley-Wiener theorem as applied in [H1, Theorem 7.3.2] shows that $w \in \mathcal{E}'$.

5. We now apply unique continuation results for $-\Delta + V$, $V \in L^\infty_{\mathrm{comp}}(\mathbb{R}^n)$ to conclude that $w \equiv 0$. $\square$

As in dimension one we want to decompose the solution (3.44) into incoming and outgoing terms. The scattering matrix will then relate these two terms.

THEOREM 3.16 (**Decomposition of free plane waves**). *We have, in the sense of distributions in $x/|x| \in \mathbb{S}^{n-1}$*

$$e^{i\lambda\langle x,\omega\rangle} \sim \frac{1}{(\lambda|x|)^{\frac{n-1}{2}}} \left(c_n^- e^{-i\lambda|x|}\delta_{-\omega}(x/|x|) + c_n^+ e^{i\lambda|x|}\delta_{\omega}(x/|x|)\right) , \tag{3.49}$$

as $|x| \to \infty$, where

$$c_n^{\pm} = (2\pi)^{\frac{n-1}{2}} e^{\mp i \frac{\pi}{4}(n-1)} .$$

More precisely for $\varphi \in C^\infty(\mathbb{S}^{n-1})$,

$$\int_{\mathbb{S}^{n-1}} e^{i\lambda r\langle\omega,\theta\rangle}\varphi(\theta)d\theta \sim \frac{1}{(\lambda r)^{\frac{n-1}{2}}} \left(c_n^- e^{-i\lambda r}\varphi(-\omega) + c_n^+ e^{i\lambda r}\varphi(\omega)\right) ,$$

$r \longrightarrow \infty$, with a full expansion in powers of r.

Proof. To prove the results we use the method of stationary phase.

1. We can assume that $\omega = (1, 0, \cdots , 0)$. Then the function $\langle\theta, \omega\rangle = \theta_1$ has two critical points on $\mathbb{S}^{n-1}$, correspoding to $\theta_1 = \pm 1$. Hence we can assume that φ is supported near the two poles $\theta_1 = \pm$ – the other contrbutions are $O((\lambda r)^{-\infty})$ as the phase is non-stationary.

2. Near the two poles we can coordinates $t \in \mathbb{R}^{n-1}$, $\theta = (\pm\sqrt{1-|t|^2}, t) \in \mathbb{S}^{n-1}$. Then, for φ supported near $\theta_1 = \pm$ we have

$$\int_{\mathbb{S}^{n-1}} e^{i\lambda r\langle\omega,\theta\langle}\varphi(\theta)d\theta) = \int_{B_{\mathbb{R}^{n-1}}(0,1)} e^{\pm i\lambda r\sqrt{1-|t|^2}}\varphi(\pm\sqrt{1-|t|^2}, t)J(t)dt ,$$

where $J(t) = 1 + \mathcal{O}(t^2)$.

3. The Hessian of the phase at $t = 0$ is given by $\mp I_{\mathbb{R}^{n-1}}$ and hence the method of stationary phase gives

$$\begin{aligned}&\int_{B_{\mathbb{R}^{n-1}}(0,1)} e^{\pm i\lambda r\sqrt{1-|t|^2}}\varphi(\pm\sqrt{1-|t|^2}, t)J(t)dt \\ &\sim \left(\frac{2\pi}{r\lambda}\right)^{\frac{n-1}{2}} e^{\mp i\frac{\pi}{4}(n-1)} \left(\varphi(\pm 1, 0) + \mathcal{O}\left(\frac{1}{r\lambda}\right)\right) ,\end{aligned}$$

with a full assymptotic expansion in powers of $(r\lambda)^{-1}$.

4. A general φ can be written as a sum of functions which are supported near $\theta_1 = \pm 1$, and in the non-stationary region. That gives the result. □

INTERPRETATION. We consider

$$\lambda^{-\frac{n-1}{2}} c_n^{\pm} \delta_{\pm\omega}(\theta)$$

as leading coefficients of the incoming ($-$) and outgoing ($+$) components of $\exp(i\lambda\langle x, \omega\rangle)$, even though that is valid only in the sense of distributions. This is an analogue of the decomposition of $\exp(\pm i\lambda x)$, $x \in \mathbb{R}$, into the incoming and outgoing components:

$$e^{\pm i\lambda x} = e^{-i\lambda|x|}(\pm x)_-^0 + e^{i\lambda|x|}(\pm x)_+^0 , \quad x \neq 0 .$$

Going back to (3.44) we see from ii) in Theorem 3.14 that w is decomposed into a sum of a plane wave and of an outgoing spherical wave u given by (3.45). The scattering matrix is defined as the operator relating the leading incoming and outgoing terms, normalized so that it is the identity when $V = 0$.

Using Theorems 3.2 and 3.16 we write the leading terms in w of (3.44) as follows:

$$\begin{gathered} e^{i\frac{\pi}{4}(n-1)} \left(\frac{2\pi}{(\lambda|x|)} \right)^{\frac{n-1}{2}} \times \\ \left(e^{-i\lambda|x|} \delta_{-\omega}\left(\frac{x}{|x|}\right) + e^{i\lambda|x|} i^{1-n} \left(\delta_\omega\left(\frac{x}{|x|}\right) + b\left(\lambda, \frac{x}{|x|}, \omega\right) \right) \right) , \\ b(\lambda, \theta, \omega) := \frac{1}{2i} \frac{\lambda^{n-2}}{(2\pi)^{n-1}} \mathcal{F}(V u(\bullet, \lambda, \omega))(\lambda\theta) . \end{gathered} \tag{3.50}$$

DEFINITION. The *absolute scattering matrix* maps the incoming terms to the outgoing terms in (3.50):

$$S_{\rm abs}(\lambda) : \delta_{-\omega}(\theta) \longmapsto i^{1-n} \left(\delta_\omega(\theta) + b(\lambda, \theta, \omega) \right) .$$

We observe that for $V = 0$ we have

$$S_{\rm abs}(\lambda) f(\theta) = i^{1-n} f(-\theta) .$$

By normalizing by this *free* absolute scattering matrix we obtain the scattering matrix:

$$S(\lambda) : \delta_\omega(\theta) \longmapsto \delta_\omega(\theta) + b(\lambda, \theta, -\omega) ,$$

where ω is given in (3.50).

We note that $V \in L^\infty(\mathbb{R}^n; \mathbb{C})$ the scattering matrix is defined away from the possible real poles of $R_V(\lambda)$ on the real axis.

We have the following following description of $S(\lambda)$:

$$
\begin{aligned}
&S(\lambda) = I + A(\lambda)\,, \\
&A(\lambda) = a_n \lambda^{n-2} E_\rho(\lambda)(I + V R_0(\lambda)\rho)^{-1} V E_\rho(\bar\lambda)^*\,, \\
&E_\rho : L^2(\mathbb{S}^{n-1}) \to L^2(\mathbb{R}^n)\,, \quad E_\rho(x,\omega) := \rho(x) e^{i\lambda\langle x,\omega\rangle}\,.
\end{aligned}
\tag{3.51}
$$

$$
A(\lambda)(\omega,\theta) =
$$

$$
a_n \lambda^{n-2} \int_{\mathbb{R}} e^{i\lambda\langle \omega-\theta, x\rangle} V(x)(1 - e^{-i\lambda\langle\omega,x\rangle} R_V(\lambda)(e^{i\lambda\langle\omega,\bullet\rangle}V)(x))dx\,,
$$

where $\theta, \omega \in \mathbb{S}^{n-1}$.

THEOREM 3.17 (Properties of the scattering matrix). *For $V \in L^\infty_{\rm comp}(\mathbb{R}^n;\mathbb{C})$ the scattering matrix, $S(\lambda)$, is meromorphic in $\mathbb{C}$ with poles of finite rank, and it satisfies*

$$
S(\lambda)^{-1} = S(-\lambda)\,, \quad \lambda \in \mathbb{C}\,. \tag{3.52}
$$

There are only finitely many poles in the closed upper half plane and for $\operatorname{Im}\lambda > 0$, $\lambda^2 \in \operatorname{Spec}(P_V)$.

When $V \in L^\infty_{\rm comp}(\mathbb{R}^n;\mathbb{R})$ then

$$
S(\lambda)^{-1} = S(\bar\lambda)^*\,, \quad \lambda \in \mathbb{C}\,. \tag{3.53}
$$

In particular, $S(\lambda)$ is unitary for $\lambda \in \mathbb{R}$ and holomorphic on $\mathbb{R}$.

In the study of resonances the following theorem provides a crucial connection. The proof will be given in Section 3.6.

THEOREM 3.18 (Multiplicities of scattering poles II). *Suppose that $S(\lambda)$ is the scattering matrix for $V \in L^\infty_{\rm comp}(\mathbb{R}^n;\mathbb{C})$, $n \geq 3$, odd.*

If we define

$$
m_S(\lambda) = \frac{1}{2\pi i} tr \oint S(\zeta)^{-1} \partial_\zeta S(\zeta) d\zeta\,, \tag{3.54}
$$

where the integral is over a positively oriented circle which includes λ and no other pole or zero of $\det S(\lambda)$, then

$$
m_S(\lambda) = m_R(\lambda) - m_R(-\lambda)\,. \tag{3.55}
$$

3.6. **Trace formulæ.**

In this section we will generalize the one dimensional trace formulæ given in Theorems 2.14 and 2.15 to the case of potential scattering in odd dimensions.

The Birman-Krein formula (Theorems 2.14 and 3.19) is valid without much change in all dimensions and for much less restrictive classes of potentials. That is not the case with the trace formulæ of Theorem 2.15 and 3.20 which cannot hold in even dimensions and are delicate for more general perturbations. Further generalizations and modifications will be discussed in Chapter 4.

We first state the main results:

THEOREM 3.19 (Birman-Krein formula II). *Suppose that* $V \in L^\infty_{\rm comp}(\mathbb{R}^n;\mathbb{R})$*, where* $n \geq 3$ *is odd.*

Then for $f \in \mathscr{S}(\mathbb{R})$ *the operator* $f(P_V) - f(P)$ *is of trace class and*

$$\begin{aligned} \operatorname{tr}\left(f(P_V) - f(P_0)\right) = \frac{1}{2\pi i}\int_0^\infty f(\lambda^2) \operatorname{tr}\left(S(\lambda)^{-1}\partial_\lambda S(\lambda)\right) d\lambda \\ + \sum_{k=1}^{K} f(E_k) + \frac{1}{2} m_R(0) f(0)\,, \end{aligned} \tag{3.56}$$

where $S(\lambda)$ *is the scattering matrix and* $E_K < \cdots < E_1 < 0$ *are the (negative) eigenvalues of* P_V*.*

The proof of this theorem will be given in a more general setting in Chapter 4.

We reiterate the remark made after Theorem 2.14 where the analogy between the counting function for eigenvalues of operators for closed systems (for instance P_V on the torus $\mathbb{R}^n/(R\mathbb{Z})^n$) and the scattering phase was made:

$$\begin{array}{ccc} \text{Counting for eigenvalues} & \longleftrightarrow & \text{Scattering phase} \\ N(\lambda) & \longleftrightarrow & \dfrac{1}{2\pi i}\log S(\lambda)\,. \end{array}$$

The next theorem is the odd dimensional analogue of Theorem 2.15 and it connects resonances with the trace of the wave group. We first observe that for

$$\varphi \in C^\infty_{\rm c}(\mathbb{R}\setminus\{0\})\,,$$

we can define the distribution

$$\varphi \longmapsto \sum_{\mathrm{Im}\lambda \leq 0} m_R(\lambda) \int_{\mathbb{R}} \varphi(t) e^{-i\lambda|t|} .$$

To see this put

$$N(r) := \sum \{ m_R(\lambda) \ : \ 0 < |\lambda| \leq r \} ,$$

so that by Theorem 3.9 we have

$$N(r) \leq C_V r^n .$$

If

$$\mathrm{supp}\varphi \subset [-R, R] \setminus \{0\}$$

then

$$\begin{aligned}
&\left| \sum_{\mathrm{Im}\lambda \leq 0} m_R(\lambda) \int_0^\infty (\varphi(t) + \varphi(-t)) e^{-i\lambda t} dt \right| \\
&\quad \leq 2R m_R(0) \sup |\varphi| + 2R \sup |\partial^N \varphi| \sum_{\mathrm{Im}\lambda < 0} m_R(\lambda) |\lambda|^{-N} \\
&\quad = 2R m_R(0) \sup |\varphi| + 2R \sup |\partial^N \varphi| \int_0^\infty r^{-N} dN(r) \\
&\quad \leq 2R m_R(0) \sup |\varphi| + 2N C_V R \sup |\partial^N \varphi| \int_1^\infty r^{-N-1} r^n dr \\
&\quad \leq C_V' R \sup_{0 \leq k \leq N} |\partial^k \varphi| ,
\end{aligned}$$

provided that $N > n$. For the integration by parts we had to assume that $0 \notin \mathrm{supp}\varphi$. However

THEOREM 3.20 (Trace formula for resonances II). *Suppose that $V \in L^\infty_{\mathrm{comp}}(\mathbb{R}^n; \mathbb{R})$, where $n \geq 3$ is odd. Then for $k \geq n$*

$$\begin{aligned}
2t^k \mathrm{tr} \left(\cos t\sqrt{P}_V - \cos t\sqrt{P}_0 \right) &= t^k \sum_{\mathrm{Im}\lambda \leq 0} m_R(\lambda) e^{-i\lambda|t|} \\
&+ \sum_{k=1}^{K} \cosh t\sqrt{-E_k} ,
\end{aligned} \tag{3.57}$$

in the sense of distributions on $\mathbb{R}$.

REMARKS. 1. As explained after Theorem 2.15 this result is an immediate consequence of the spectral theorem for self-adjoint operators with discrete spectra. It is quite remarkable that the same theorem holds (in odd dimensions, and for compactly supported perturbations) in an exactly the same form for resonances.

2. The factor of t^k in (3.57) is needed as there are many possible extensions of the distribution $\sum_{\mathrm{Im}\lambda\leq 0} m_R(\lambda)\exp(-i|t|\lambda)$ from $\mathbb{R}\setminus\{0\}$ to $\mathbb{R}$. There is no known analogue of the improved formula given in Theorem 2.16.

3. The trace formula (3.57) can be considered as an abstract consequence of the Birman-Krein formula and of the Hadamard factorization the scattering determinant, $\det S(\lambda)$, as a meromorphic function – see Theorem 4.1 below. In Chapter 4 once we establish the Birman-Krein formula and the properties of $\det S(\lambda)$ we will simply quote the proof given later in this section.

The next result is a very useful as it relates the determinant of the scattering matrix to the determinant of an operator acting on $L^2(\mathbb{R}^n)$.

THEOREM 3.21 (Trace identities). *Suppose that*

$$V \in L^\infty_{\mathrm{comp}}(\mathbb{R}^n;\mathbb{C})\,, \quad n\geq 3\,,\ \textit{odd,}$$

and that $\rho\in C^\infty_{\mathrm{c}}(\mathbb{R}^n)$ *is equal to one on* $\mathrm{supp}V$.

Let

$$T(\lambda) := (I - VR_0(\lambda)\rho)^{-1}(V(R_0(\lambda)-R_0(-\lambda))\rho)\,.$$

Then $T(\lambda)$ *is a strace class operator and*

$$\det S(\lambda) = \det(I-T(\lambda))\,. \tag{3.58}$$

Proof. 1. The operator $T(\lambda)$ is of trace class since

$$\rho R_0(\lambda) - R_0(-\lambda)\rho : L^2(\mathbb{R}^n) \longrightarrow H^k([-R,R])\,,$$

for any k, provided that $\rho\in C^\infty(\mathbb{R}^n)$, $\mathrm{supp}\rho\subset B(0,R)$.

2. We will prove the formula for $\lambda\in\mathbb{R}$. For that we first write $S(\lambda) = I - A(\lambda)$ where, using (3.28),

$$\begin{aligned} A(\lambda) &= c_n\lambda^{n-2}E_\rho(\lambda)(I - VR_V(\lambda)\rho)VE_\rho(\lambda)^* \\ &= c_n\lambda^{n-2}E_\rho(\lambda)(I + VR_0(\lambda)\rho)^{-1}VE_\rho(\lambda)^*\,. \end{aligned}$$

3. To prove (3.58) all we need to show is that for all $k\in\mathbb{N}$

$$\mathrm{tr}T(\lambda)^k = \mathrm{tr}A(\lambda)^k\,. \tag{3.59}$$

In fact, (3.59) shows for t small (so that the log can be defined),

$$\begin{aligned}\log\det(I - tT(\lambda) &= \operatorname{tr}\log(I - tT(\lambda))\\ &= \operatorname{tr}\log(I - tA(\lambda))\\ &= \log\det(I - tA(\lambda))\,.\end{aligned}$$

It follows that $\det(I - tA(\lambda)) = \det(I - tT(\lambda))$ for small values of t, and by analytic continuation in t, for $t = 1$.

4. To establish (3.59) we use ($\lambda \in \mathbb{R}$)

$$\rho(R_0(\lambda) - R_0(-\lambda))\rho = c_n\lambda^{n-2}E_\rho(\lambda)^*E_\rho(\lambda)$$

in the definition of $T(\lambda)$:

$$T(\lambda) = c_n\lambda^{n-2}(I - VR_0(\lambda)\rho)^{-1}VE_\rho(\lambda)^*E_\rho(\lambda)\,.$$

Let $A = E_\rho(\lambda)$, $B = (I - VR_0(\lambda)\rho)^{-1}V$, $C = E_\rho(\lambda)^*$ so that A and C are trace class operators (between different Hilbert spaces):

$$A : H_1 \to H_2, \quad B : H_1 \to H_1\,, \quad C : H_2 \to H_1\,,$$

and B is a bounded operator. Cyclicity of trace shows that

$$\begin{aligned}\operatorname{tr}_{H_1}(ABC)^n &= \operatorname{tr}_{H_1}A(BCA)^{n-1}BC\\ &= \operatorname{tr}_{H_2}BCA(BCA)^{n-1}\\ &= \operatorname{tr}_{H_2}(BCA)^n\,.\end{aligned}$$

This gives (3.59). $\square$

As the first consequence of Theorem 3.21 we prove the relation between the poles of $S(\lambda)$ and of $R_V(\lambda)$ announced in Theorem 3.18:

$$m_S(\lambda) = m_R(\lambda) - m_R(-\lambda)\,.$$

Proof of Theorem 3.18. 1. The formula (3.58) implies the following inequality involving regularized determinants, $\det_p$, is defined in (B.21): for any $p > (n+1)/2$,

$$\begin{aligned}&\det S(\lambda) = \frac{\det_p(I + VR_0(-\lambda)\rho)}{\det_p(I + VR_0(\lambda)\rho)}e^{g_p(\lambda)}\\ &g_p(\lambda) := \sum_{\ell=1}^{p-1}\frac{(-1)^\ell}{\ell}\operatorname{tr}\left((\rho R_0(\lambda)V)^\ell - (\rho R_0(-\lambda)V)^\ell\right)\,.\end{aligned} \tag{3.60}$$

2. We now apply Theorem 3.8. Since $\exp g_p(\lambda)$ does not contribute to the multiplicity of the right hand side, the theorem follows. $\square$.

The next application provides a Hadamard factorization of the scattering matrix:

THEOREM 3.22 (**Factorization of the scattering matrix I).** *Suppose that $V \in L^\infty(\mathbb{R}^n; \mathbb{C})$ where $n \geq 1$ is odd. Then*

$$\det S(\lambda) = e^{g(\lambda)} \frac{P(\lambda)}{P(-\lambda)},$$

$$P(\lambda) := \prod E_n(\lambda/\mu)^{m_R(\mu)}, \quad E_n(z) := (1-z)e^{z+z/2+\cdots+z^n/n}, \tag{3.61}$$

$$g(\lambda) = a_n \lambda^n + a_{n-2}\lambda^{n-2} + \cdots + a_1 \lambda.$$

When $V \in L^\infty_{\rm comp}(\mathbb{R}^n; \mathbb{R})$, that is, when $S(\lambda)$ is unitary for $\lambda \in \mathbb{R}$, $a_n \in i\mathbb{R}$.

Proof. 1. Theorem 3.18 already shows that (3.61) holds with g given an entire function. Since $S(\lambda) = S(-\lambda)^{-1}$ we see that g has to be odd. Hence all we need to show is that g is a polynomial of degree at most n.

2. We will first establish two preliminary bounds

$$|\det S(\lambda)| \leq \begin{cases} C \exp C|\lambda|^n, & \operatorname{Im}\lambda \geq 0, |\lambda| > C, \\ C\exp C|\lambda|^{2n^2}, & \lambda \notin \bigcup_{m_H(\mu)>0} D(\mu, \langle\mu\rangle^{-n-\epsilon}), \end{cases} \tag{3.62}$$

where $m_H(R)$ was defined in (3.27). In view of (3.58) this will follow from estimates on the characteristic values in the spirit of the proof of Theorem 3.9.

3. To apply estimates on characteristic values we use (3.33) to write

$$T(\lambda) = c_n (I + VR_0(\lambda)\rho)^{-1} V E_\rho(\bar\lambda)^* E_\rho(\lambda).$$

From (3.10) we see that

$$\|(I + VR_0(\lambda)\rho)^{-1}\| \leq C, \quad \operatorname{Im}\lambda \geq 0, \quad |\lambda| \geq C.$$

Hence in the same range of λ's we have

$$s_j(T(\lambda)) \leq C|\lambda|^{n-2} \|E_\rho(\bar\lambda)^*\| s_j(E_\rho(\lambda)).$$

Applying (3.36) we obtain (with a different constant C)

$$s_j(T(\lambda)) \leq C \exp\left(C|\lambda| - j^{\frac{1}{n-1}}/C\right), \quad \operatorname{Im}\lambda \geq 0, \quad |\lambda| \geq C.$$

The Weyl inequality can now be applied as in part 5 of the proof of Theorem 3.9 gives

$$
\begin{aligned}
|\det(I - T(\lambda))| &\leq \prod_{j=0}^{\infty}(1 + s_j(T(\lambda))) \\
&\leq \prod_{j \leq (2C^2|\lambda|)^{n-1}} (1 + e^{C|\lambda|}) \exp \sum_{j \geq 1} e^{-j^{1/n-1}/(2C)} \\
&\leq C' \exp C'|\lambda|^n ,
\end{aligned}
$$

and this proves the first part of (3.62).

4. We now consider the case of λ outside of a union of discs containing reasonances. First we note that for there exists a sequence $r_k \to \infty$, such that

$$
\forall\, k\,, \quad \partial D(0, r_k) \cap \bigcup_{m_H(\mu)>0} D(\mu, \langle\mu\rangle^{-n-\epsilon}) = \emptyset\,, \tag{3.63}
$$

which follows from the fact

$$
\sum_{m_H(\mu)>0} \langle\mu\rangle^{-n-\epsilon} < \infty\,,
$$

which is in turn implied by (3.29).

To estimate $\|(I + VR_0(\lambda)\rho)^{-1}\|$ away from resonances we use (B.23) with $p = n+1$:

$$
\|(I + VR_0(\lambda)\rho)^{-1}\| \leq \frac{G(\lambda)}{H(\lambda)}\,,
$$

where

$$
G(\lambda) := \prod_{j=0}^{\infty}(1 + s_j(VR_0(\lambda))^{n+1})\,, \quad H(\lambda) := \det(I - (VR_0(\lambda)\rho)^{n+1})\,.
$$

Theorem 3.9 shows that $H(\lambda)$ is an entire function of order n, and its proof shows that

$$
G(\lambda) \leq C \exp(C|\lambda|^n)\,.
$$

The minimum modulus theorem for entire functions of order n (see (D.8)) shows that

$$
|H(\lambda)| \geq \exp(-C_\epsilon|\lambda|^{n+\epsilon})\,, \quad \lambda \notin \bigcup_{m_H(\mu)>0} D(\mu, \langle\mu\rangle^{-n-\epsilon})\,.
$$

Hence for λ's in the same set we obtain

$$
\|(I + VR_0(\lambda)\rho)^{-1}\| \leq C \exp(C|\lambda|^{2n+1})\,.
$$

Returning to singular values of $T(\lambda)$ this gives

$$s_j(T(\lambda)) \leq C \exp\left(C|\lambda|^{2n+1} - j^{\frac{1}{n-1}}/C\right), \quad \lambda \notin \bigcup_{m_H(\mu)>0} D(\mu, \langle\mu\rangle^{-n-\epsilon}).$$

The same argument as before gives the second part of (3.62).

5. We now recall the estimates on Weierstrass products (see (D.7) and (D.8) in Section D):

$$e^{-C_\epsilon|\lambda|^{n+\epsilon}} \leq |P(\lambda)| \leq e^{C_\epsilon|\lambda|^{n+\epsilon}}, \quad \lambda \notin \bigcup_{m_H(\mu)>0} D(\mu, \langle\mu\rangle^{-n-\epsilon}).$$

Hence in the same set of λ's

$$\begin{aligned} |\exp(g(\lambda))| &= |\exp(-g(-\lambda)| \\ &= |\det S(-\lambda)| \frac{|P(-\lambda)|}{|P(\lambda)|} \\ &\leq C \exp(C|\lambda|^{2n^2} + C|\lambda|^{n+\epsilon}) \\ &\leq C \exp C|\lambda|^{2n^2}. \end{aligned} \tag{3.64}$$

We can now use this on circles of radius r_k satisfying (3.63) so that the maximum principle show that the above estimate holds everywhere.

Hence,

$$\mathrm{Re} g(\lambda) \leq C|\lambda|^{2n^2}.$$

An application of the Borel-Carathéodory theorem (D.3) gives

$$|g(\lambda)| \leq C|\lambda|^{2n^2},$$

which implies that g is a polynomial.

6. It remains to show that $g(\lambda)$ is a polynomial of degree n. For that we apply the same strategy as in (3.64) but for $\mathrm{Im}\lambda \leq 0$, $|\lambda| \geq C$ so that we can use the first estimate in (3.62). That gives

$$\mathrm{Re} g(\lambda) \leq C_\epsilon \|\lambda\|^{n+\epsilon}, \quad \mathrm{Im}\lambda \leq 0, \quad |\lambda| \geq C.$$

For $n \geq 2$ any polynomial satisfying this bound has to have degree at most n.

7. The last statement about the polynomial g when V is real valued comes from the unitarity of the scattering matrix. $\square$

We can now give the proof of Theorem 3.20.

Proof of Theorem 3.20. We follow the proof of (2.15) closely with modifications due to the change in the growth of resonances. Theorem 3.22

is essential new component. We again make the simplifying assumption that there are no eigenvalues and that 0 is not a resonance.

1. Let us first consider the statement on $\mathbb{R} \setminus \{0\}$. Just as in the proof of Theorem 2.15 (3.57) is equivalent to the following statement: for $\varphi \in C_{\rm c}^\infty((0,\infty))$

$$\begin{gathered} f(P_V) - f(P_0) = \sum_{\operatorname{Im}\lambda<0} \widehat{\varphi}(\lambda) m_R(\lambda) + (m_R(0)-1)\widehat{\varphi}(0)\,, \\ f(z) := \widehat{\varphi}(\sqrt{z}) + \widehat{\varphi}(-\sqrt{z})\,, \quad f \in \mathscr{S}(\mathbb{R})\,. \end{gathered} \tag{3.65}$$

Theorem3.19 and evennes of $\partial_\lambda \log\det S(\lambda)$ reduce the proof to showing that

$$\frac{1}{2\pi i}\int_{-\infty}^{\infty} \widehat{\varphi}(\lambda)\partial_\lambda(\log\det S(\lambda))d\lambda = \sum_{\operatorname{Im}\lambda<0} \widehat{\varphi}(\lambda) m_R(\lambda)\,. \tag{3.66}$$

2. We will now use the factorization (3.61). A calculation based on the formula for E_n and P given there shows that

$$\partial_\lambda^{n+1} \log P(\lambda) = -\sum_{\operatorname{Im}\mu<0} \frac{m_R(\mu)}{(\lambda-\mu)^{n+1}}\,,$$

where the bound (3.29) gives convergence.

Hence (3.61) gives

$$\partial_\lambda^{n+1}(\log\det S(\lambda)) = \sum_{\operatorname{Im}\mu<0} \frac{m_R(\mu)}{(\lambda-\mu)^{n+1}} - \sum_{\operatorname{Im}\mu<0} \frac{m_R(\mu)}{(\lambda+\mu)^{n+1}}\,.$$

3. Define $g \in \mathcal{S}$ by $\partial^n g(\lambda) = \widehat{\varphi}(\lambda)$. This is possible as φ supported in $(0,\infty)$:

$$(\lambda) = i^n \widehat{\varphi/t^n}(\lambda)\,. \tag{3.67}$$

Now,

$$\begin{aligned} \int_{\mathbb{R}} \widehat{\varphi}(\lambda)\partial_\lambda(\log\det S(\lambda))d\lambda &= -\int_{\mathbb{R}} g(\lambda)\partial_\lambda^{n+1}(\log\det S(\lambda))d\lambda \\ &= \sum_{\pm} \pm \sum_{\operatorname{Im}\mu<0} \int_{\mathbb{R}} \frac{m_R(\mu)}{(\lambda\mp\mu)^{n+1}} g(\lambda)d\lambda \\ &= 2\pi i \sum_{\operatorname{Im}\mu<0} m_R(\mu)\partial^n g(\mu) \\ &= 2\pi i \sum_{\operatorname{Im}\mu<0} m_R(\mu)\widehat{\varphi}(\mu)\,, \end{aligned}$$

where we deformed the contour in the integral using the fact that

$$\widehat{\varphi} \in C_{\mathrm{c}}^{\infty}((0,\infty)) \implies |\partial^n g(\lambda)| = \mathcal{O}((1+|\lambda|)^{-\infty}), \quad \text{for } \mathrm{Im}\lambda \leq 0,$$

Since this gives (2.71) the proof is complete when we only consider testing against functions in $C_{\mathrm{c}}^{\infty}(\mathbb{R} \setminus \{0\})$.

4. To obtain the more general statement we need to consider the argument above with $\varphi(t)$ replaced by $t_+^k \varphi$ where $k \geq n$. In that case g defined by (3.67) satisfies

$$|\partial^n g(\lambda)| = \mathcal{O}((1+|\lambda|)^{-n-1}), \quad \text{for } \mathrm{Im}\lambda \leq 0,$$

but that is enough to justify the arguments above. □

The final application of Theorem 3.21 concerns asymptotics of the scattering phase. It is of intrinsic interest but it will also play an important rôle in the next section.

THEOREM 3.23 (**Asymptotics of the scattering phase**). *Suppose that* $V \in L^{\infty}_{\mathrm{comp}}(\mathbb{R}^n, \mathbb{C})$ *where* $n \geq 1$ *is odd. Define the* scattering phase

$$\sigma(\lambda) := \frac{1}{2\pi i} \log \det S(\lambda), \quad \sigma(0) = 0. \tag{3.68}$$

Then there exists a sequence $c_k(V)$ *such that*

$$\sigma(\lambda) \sim \sum_{k=1}^{\infty} c_k(V) \lambda^{n-2k}, \quad \lambda \longrightarrow \infty, \tag{3.69}$$

where

$$c_1(V) = c_n \int_{\mathbb{R}^n} V(x) dx, \quad c_2(V) = c_n' \int_{\mathbb{R}^n} V(x)^2 dx. \tag{3.70}$$

Proof. We will use the determinant formula (3.60). □

REMARK. The method of proof is not the best for finding the coefficients $c_k(V)$. The now classical connection to the heat or wave kernels provides more efficient algorithms – see [Gu] and references given there.

3.7. **Existence of resonances for real potentials.**

In Theorem 2.11 we proved that any complex valued compactly supported potential in one dimension has infinitely many resonances. In Section 3.4 earlier in this chapter we have shown that there exist complex valued compacty supported potentials in higher dimensions with *no* resonances.

In this section we will prove the following

THEOREM 3.24 (Existence of resonances). *Suppose that* $V \in L^\infty_{\mathrm{comp}}(\mathbb{R}^n, \mathbb{R})$*,* $n \geq 3$*, odd. Then*

$$\sum_{\lambda \in \mathbb{C}} m_R(\lambda) = \infty \,,$$

that is V *has infinitely many scattering resonances.*

Proof. 1. We will first assume that there are no non-zero resonances. Then Theorem 3.22 implies that

$$\begin{aligned} \sigma(\lambda) &:= \frac{1}{2\pi i} \log \det S(\lambda) \\ &= b_n \lambda^n + b_{n-2} \lambda^{n-2} + \cdots + b_1 \lambda \,, \quad b_j \in \mathbb{R} \,. \end{aligned} \tag{3.71}$$

Comparison with (3.69) and (3.70) shows that

$$b_n = 0 \,, \quad b_{n-2} = c_n \int_{\mathbb{R}^n} V(x) dx \,, \quad b_{n-4} = c_n' \int_{\mathbb{R}^n} V(x)^2 dx \neq 0 \,.$$

This gives an immediate contradiction when $n = 3$ as then $g(\lambda) = b_1 \lambda$.

2. To obtain a contradiction for $n > 3$. We consider the behaviour of $\sigma(\lambda)$ as $\lambda \to 0$. The formula (3.51) shows that if $R_V(\lambda)$ is holomorphic near 0, which it is due to our assumption that there are no resonances, then

$$\| A(\lambda) \|_{\mathcal{L}_1} = \mathcal{O}(\lambda^{n-2}) \,, \quad \lambda \longrightarrow 0 \,.$$

Using this we see that near 0,

$$\begin{aligned} 2\pi i \sigma(\lambda) &= \log \det (I + A(\lambda)) = \operatorname{tr} \log (I + A(\lambda)) \\ &= \mathcal{O}(\| A(\lambda) \|_{\mathcal{L}_1}) = \mathcal{O}(\lambda^{n-2}) \,. \end{aligned}$$

Comparing this with (3.71) we see that

$$\sigma(\lambda) = b_{n-2} \lambda^{n-2} \,.$$

But this contradicts the fact that $b_{n-4} \neq 0$.

3. It remains to show that the number of resonances is infinite. Again we proceed by contradiction. Suppose that there exists a finite number of non-zero resonances. Noting that we took $\sigma(0) = 0$, suppose that

$$\begin{aligned}\sigma(\lambda) &= b(\lambda) + \frac{1}{2\pi i}\log\left((-1)^K \prod_{k=1}^{K} \frac{\lambda + \mu_k}{\lambda - \mu_k}\right) \\ &= b_{n-2}\lambda^{n-2} + \cdots + b_1\lambda + \frac{K}{2} + \mathcal{O}(1/\lambda)\,,\end{aligned}$$

as $\lambda \to \infty$. This contradicts the asymptotic expansion (B.14) as it has no even terms. $\square$

3.8. **Sources and further reading.**

For a discussion of the threshold behaviour for non-compactly supported potentials see [Je-Ne] and references given there.

The proof of Theorem 3.9 is based on ideas of Melrose who proved the bound

$$\sum\{m_R(\lambda) \;:\; |\lambda| \leq r\} \leq C_V r^{n+1}\,.$$

The optimal bound (3.29) was proved in [Z3]. Our presentation uses a substantial simplification of the argument due to Vodev [Vo] – see Chapter 4 for further applications of these methods.

The class of examples in Theorem 3.11 was constructed by Christiansen [Ch].

The trace identity in Theorem 3.21 was proved by Buslaev in [Bus]. Theorem 3.23 and further references can be found in [Gu].

That a potential in any odd dimension has infinitely many resonances was proved in [SaB-Zw] but the method there was less direct. There have been many improvement since. Christiansen and Hislop [Ch-Hi] proved that for a generic $L^\infty_{\rm com}(\mathbb{R}^n, \mathbb{R})$ (or $C^\infty_{\rm c}(\mathbb{R}^n, \mathbb{R})$) potential the exponent n in the polynomial bound (3.29) is optimal. That relied on the existence of a lower bound given by (3.38) and is also true for generic *complex valued* potentials. That paper can be consulted for intermediate results on lower bounds.

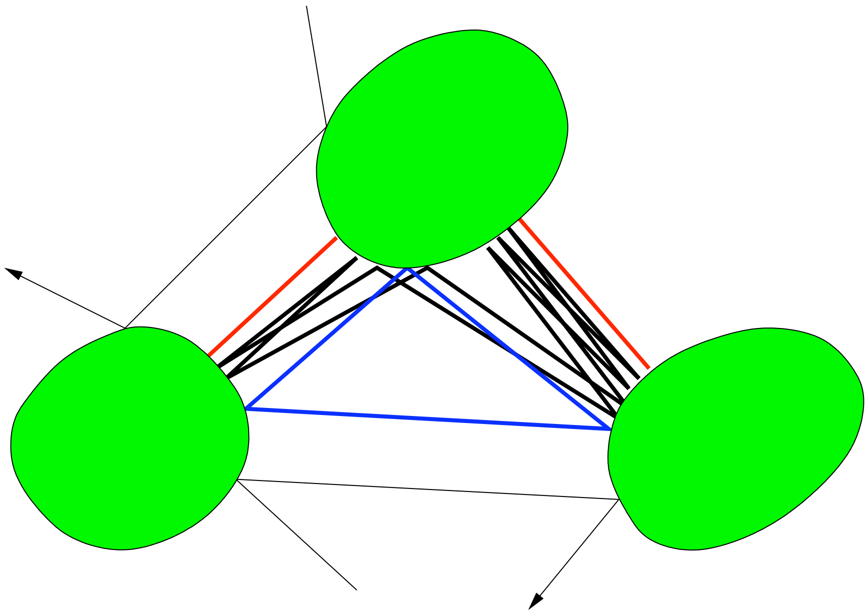

FIGURE 14. An example of trapped trajectories in obstacle scattering.

4. Black box scattering in $\mathbb{R}^n$

In Sections 2 and 3 we have studied general properties of resonances in scattering by compactly supported potential. More general compactly supported perturbations include metric perturbations, and obstacle scattering. They offer many new interesting and relevant physical features such a presence of trapping – see Fig. 14

For general results of the type presented in the last two sections it is convenient to replace a specific perturbation by an abstractly defined *black box* perturbation.

The following table shows the basic differences and analogies in the case when n is odd.

Here P denotes the operator equal to $-\Delta$ outside $B(0, R_0)$ – see Section 4.1 for precise assumption. The operator P is assumed to act on a Hilbert space $\mathcal{H}$ with an orthogonal decomposition $\mathcal{H}_{R_0} \oplus L^2(\mathbb{R}^n \setminus B(0, R_0)$ and $\mathbb{1}_{B(0,R_0}$ denotes the orthogonal projection onto the first component – the *black box*.

$-\Delta + V$	Black Box
Meromorphy of the resolvent $R_V(\lambda) : L^2_{\rm comp} \to L^2_{\rm loc}$	OK when $\mathbb{1}_{B(0,R_0)}(P-i)^{-1}$ is compact, $R(\lambda) : \mathcal{H}_{\rm comp} \to \mathcal{H}_{\rm loc}$, meromorphic
Upper bound on the number of resonances, $N(r) \leq Cr^n$	Upper bounds in terms of eigenvalues of a reference operator: $N(r) \leq CM(Cr)$, where $M(r)$ is the counting function of eigenvalues of P acting on $\mathcal{H}_{R_0} \oplus L^2(\mathbb{R}^n/((R_0+1)\mathbb{Z})^n \setminus B(0,R_0))$.
Trace formula for resonances	OK when for some k $\mathbb{1}_{B(0,R_0)}(P-i)^{-k} \in \mathcal{L}_1(\mathcal{H},\mathcal{H})$
Pole free region	Need geometric assumptions e.g. $P = -\Delta_g$ and all the geodesics of metric g escape to infinity
Resonance expansions of wave	Delicate when there are no large pole free regions

4.1. **General assumptions.**

4.2. **Meromorphic continuation.**

4.3. **Global upper bounds on the number of resonances.**

4.4. **Scattering matrix for general compactly supported perturbations.**

4.5. **Trace formulæ in black box scattering.**

THEOREM 4.1 (**Factorization of the scattering matrix II**). *Suppose that P satisfies the general assumptions of Section 4.1 and that n is odd. Then*

$$\det S(\lambda) = e^{g(\lambda)} \frac{P(\lambda)}{P(-\lambda)} \, ,$$

$$P(\lambda) := \prod E_n(\lambda/\mu)^{m_R(\mu)} \, , \quad E_n(z) := (1-z)e^{z+z/2+\cdots+z^n/n} \, , \tag{4.1}$$

$$g(\lambda) = a_n \lambda^n + a_{n-2}\lambda^{n-2} + \cdots + a_1 \lambda \, .$$

When $V \in L^\infty_{\rm comp}(\mathbb{R}^n;\mathbb{R})$, that is, when $S(\lambda)$ is unitary for $\lambda \in \mathbb{R}$, $a_n \in i\mathbb{R}$.

4.6. **Sources and further reading.**

The black formalism was introduced in [S-Z1].

5. **The method of complex scaling**

For the moment we only provide a brief review borrowed from [S-Z10]. In the simple but instructive setting of dimension one the method was described in Section 2.6. **Detailed discussion of the higher dimensional method will be presented later.**

Since we will use the method for genera semiclassical operators our assumptions on the operator are made in that setting.

We now state the general assumptions on the operator P. The simplest case to keep in mind is

$$P = -h^2\Delta + V(x) - 1\,, \quad V \in C^\infty_{\rm c}(\mathbb{R}^n)\,.$$

In general we consider

$$P(h) \in \Psi^2(X)\,, \quad P(h) = P(h)^*\,,$$

where the calculus of semiclassical pseudodifferential operators is reviewed in Appendix E.

$$\begin{gathered} P(h) = p^w(x, hD) + hp_1^w(x, hD; h)\,, \quad p_1 \in S^2(T^*X)\,, \\ |\xi| \geq C \implies p(x,\xi) \geq \langle \xi \rangle^2/C\,, \quad p = 0 \implies dp \neq 0\,, \\ \exists\, R,\ \forall\, u \in C^\infty(X \setminus B(0,R))\,, \quad P(h)u(x) = Q(h)u(x)\,, \end{gathered} \tag{5.1}$$

where

$$Q(h) = \sum_{|\alpha|\leq 2} a_\alpha(x;h)(hD_x)^\alpha\,,$$

with $a_\alpha(x;h) = a_\alpha(x)$ independent of h for $|\alpha| = 2$, $a_\alpha(x;h) \in S(\mathbb{R}^n)$ uniformly bounded with respect to h (here $S(\mathbb{R}^n)$ denotes the space of C^∞ functions on $\mathbb{R}^n$ with bounded derivatives of all orders – see Appendix E), and

(5.2)

$$\sum_{|\alpha|=2} a_\alpha(x)\xi^\alpha \geq (1/c)|\xi|^2, \ \ \forall \xi \in \mathbb{R}^n\,, \text{ for some constant } c > 0,$$

$$\sum_{|\alpha|\leq 2} a_\alpha(x;h)\xi^\alpha \longrightarrow \xi^2 - 1 \text{ uniformly with respect to } h \text{ as } |x| \to \infty.$$

We also need the following analyticity assumption in a neighbourhood of infinity: there exist $\theta \in [0,\pi)$, $\epsilon > 0$ and $R \geq R_0$ such that the coefficients $a_\alpha(x;h)$ of $Q(h)$ extend holomorphically in x to

$$\{r\omega : \omega \in \mathbb{C}^n\,, \ \ \operatorname{dist}(\omega, \mathbf{S}^n) < \epsilon\,,\ r \in \mathbb{C}\,,\ |r| > R\,,\ \arg r \in [-\epsilon, \theta_0 + \epsilon)\}\,,$$

with (5.2) valid also in this larger set of x's.

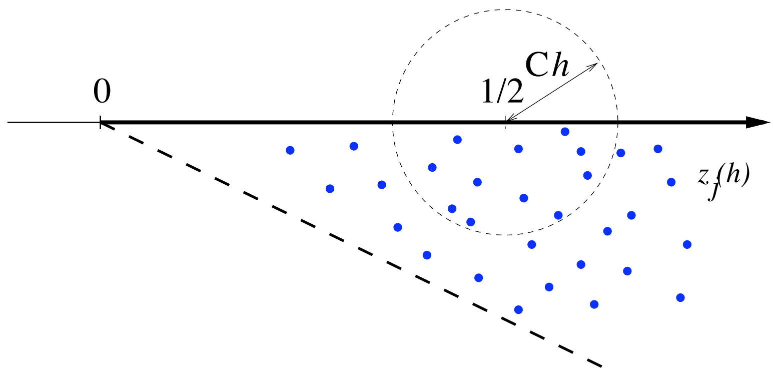

FIGURE 15. The resonances as eigenvalues the scaled operator P_θ.

We very briefly recall the *complex scaling* procedure developed in [Sj-2]. It follows a long tradition of the complex scaling method – see [Ag-Co],[Ba-Co],[Si1] for the original ideas, and [S-Z1] for the approach used here for compactly supported perturbations.

Let $\Gamma_\theta \subset \mathbb{C}^n$ be a totally real contour with the following properties:

$$\begin{gathered} \Gamma_\theta \cap B_{\mathbb{C}^n}(0, R_0) = B_{\mathbb{R}^n}(0, R_0)\,, \\ \Gamma_\theta \cap \mathbb{C}^n \setminus B_{\mathbb{C}^n}(0, 2R_0) = e^{i\theta}\mathbb{R}^n \cap \mathbb{C}^n \setminus B_{\mathbb{C}^n}(0, 2R_0)\,, \\ \Gamma_\theta = \{x + i f_\theta(x) \ : \ x \in \mathbb{R}^n\}\,, \quad \partial_x^\alpha f_\theta(x) = \mathcal{O}_\alpha(\theta)\,. \end{gathered} \tag{5.3}$$

The contour can be considered as a deformation of the manifold X as nothing is being done in the compact region. The operator P defines a dilated operator:

$$P_\theta \stackrel{\text{def}}{=} P|_{\Gamma_\theta}\,, \quad P_\theta u = \widetilde{P}(\tilde{u})|_{\Gamma_\theta}\,,$$

where $\widetilde{P}$ is the holomorphic continuation of the operator P, and $\tilde{u}$ is an almost analytic extension of $u \in C^\infty_{\rm c}(\Gamma_\theta)$ (here we are only concerned with $\Gamma_\theta \cap B_{\mathbb{C}^n}(0, R_0)$).

For θ fixed, the scaled operator, P_θ, is uniformly elliptic in $\Psi^{0,2}(X)$ outside a compact set and hence the resolvent, $(P_\theta - z)^{-1}$, is meromorphic for $z \in D(0, 1/C)$. We can also take θ to be h dependent and the same statement holds for $z \in D(0, \theta/C)$. The spectrum of P_θ in $z \in D(0, \theta/C)$ is independent of θ and consists of quantum resonances of P which are defined as the poles of the meromorphic continuation of

$$(P - z)^{-1} \ : \ C^\infty_{\rm c}(X) \longrightarrow C^\infty(X)\,.$$

In fact, that is one of the ways of defining resonances, and in this paper we will be estimating the number of eigenvalues of P_θ.

6. Perturbation theory for resonances

6.1. Generic simplicity of resonances.

6.2. Fermi golden rule.

7. **Resolvent estimates in semiclassical scattering**

In this chapter we will consider resonances close to the real axis in semiclassical scattering. The example to keep in mind is

$$P = -h^2\Delta + V(x) \,, \quad V \in C^\infty_{\rm c}(\mathbb{R}^n, \mathbb{R}) \,, \tag{7.1}$$

but our assumptions allow any operator $P(x, hD)$ to which the complex scaling method of Chapter 5 applies.

We denote

$$\mathrm{Res}(P(h)) \subset \{\mathrm{Im} z \leq 0\}$$

the set of resonances that is the set of poles of the meromorphic continuations of $(P-z)^{-1} : C^\infty_{\rm c}(\mathbb{R}^n) \to C^\infty(\mathbb{R}^n)$. We denote this continuation

$$R(z,h) : C^\infty_{\rm c}(\mathbb{R}^n) \to C^\infty(\mathbb{R}^n) \,, \quad R(z,h) = (P-z)^{-1} \,, \quad \mathrm{Im} z > 0 \,.$$

7.1. **Classical scattering theory.**

We start by presenting some basic concepts of classical scattering theory. It concern the flow of the Hamiltonian $p = p(x, \xi)$ which we assume satisfies the assumptions (5.1) and (5.2).

The classical flow is defined using the Hamilton vector field

$$H_p = \sum_{j=1}^n \frac{\partial}{\partial \xi_j}\frac{\partial}{\partial x_j} - \frac{\partial}{\partial x_j}\frac{\partial}{\partial \xi_j} \,,$$

as

$$\Phi^t(x,\xi) = \exp(tH_p)(x,\xi) \,,$$

see [EZ, Section 2.3]. For the basic example of

$$p(x,\xi) = \xi^2 + V(x) \,, \quad V \in C^\infty_{\rm c}(\mathbb{R}^n;\mathbb{R}) \,,$$

the flow is given by solving Newton's equations

$$\dot{x} = 2\xi \,, \quad \dot{\xi} = -dV(x) \,,$$

and it is given by the free Euclidean flow

$$\dot{x} = 2\xi \,, \quad \dot{x} = 0 \,,$$

outside a compact set.

We think of (x, ξ), the position and momentum, to lie in the cotangent bundle of $\mathbb{R}^n$,

$$(x,\xi) \in T^*\mathbb{R}^n \simeq \mathbb{R}^n \times \mathbb{R}^n \,,$$

and we denote

$$\pi : T^*\mathbb{R}^n \longrightarrow \mathbb{R}^n \,, \quad \pi(x,\xi) = x \,,$$

the projection to the base.

The incoming and outgoing tails of the flow at energy E are defined as

$$\Gamma_E^\pm = \{(x,\xi) \in p^{-1}(E) \ : \ |\pi(\Phi^t(x,\xi)| \not\to \infty \, , \ t \to \mp\infty\} \, .$$

We think of $\Gamma_E^\pm$ is forward $(-)$ and backward $(-)$ *trapped sets* at energy E.

For $J \subset \mathbb{R}$ we also define

$$\Gamma_J^\pm := \bigcup_{E \in J} \Gamma_E^\pm \, . \tag{7.2}$$

Strictly speaking $\Gamma_E^\pm$ should be denoted $\Gamma_{\{E\}}^\pm$ but we allow ourselves this notational lapse.

The *trapped set* is defined as

$$K_E := \Gamma_E^+ \cap \Gamma_E^- \, , \tag{7.3}$$

with K_J, $J \subset \mathbb{R}$ defined analogously.

THEOREM 7.1 (**Properties of trapped sets).** *The* $\Gamma_J^\pm$ *and* K_J *defined in* (7.2) *and* (7.3) *have the following properties:*

i) If $J \subset \mathbb{R}$ *is closed then* $\Gamma_J^\pm$ *are closed.*

ii) If $K_E = \emptyset$ *then for some neighbourhood of* E*,* $J = (E - \delta, E + \delta)$*,* $\delta > 0$*,* $K_J = \emptyset$*.*

iii) If $\Gamma_E^+ \neq \emptyset$ *or if* $\Gamma_E^- \neq \emptyset$ *then* $K_E \neq \emptyset$*.*

iv) Suppose that $J = [a, b]$*, and denote by* m *the canonical measure on* $T^*\mathbb{R}^n$*. Then*

$$m(\Gamma_J^\pm \setminus K_J) = 0 \, .$$

v) Suppose that $dp|_{p^{-1}(E)} \neq 0$ *and let* $\mathcal{L}_E$ *denote the Liouville measure on* $p^{-1}(E)$*.*

$$\mathcal{L}_E(\Gamma_E^\pm \setminus K_E) = 0 \, .$$

7.2. **Non-trapping estimates; resonance free regions.**

When there is no trapping we have very good estimates on the resolvent:

THEOREM 7.2 (Resonance free regions for non-trapping perturbations). *Suppose that P satisfies general assumptions of Chapter 5 and that for some $E >$ the trapped set at energy E defined by* (7.3) *is empty:*

$$K_E = \emptyset\,.$$

Then there exists $\delta > 0$ such that for each $M > 0$ there exists $h_M > 0$ so that

$$\operatorname{Res}(P(h)) \cap ([E-\delta, E+\delta] - i[0, Mh\log(1/h)]) = \emptyset\,, \tag{7.4}$$

$0 < h < h_M$.

In addition we have a bound on the truncated resolvent: for $\chi \in C^\infty_{\rm c}(\mathbb{R}^n)$,

$$\|\chi R(z,h)\chi\|_{L^2\to L^2} \leq \frac{C\exp(C\operatorname{Im} z/h)}{h}\,, \tag{7.5}$$

$z \in [E-\delta, E+\delta] - i[0, Mh\log(1/h)]$, $0 < h < h_M$.

Proof. 1. Let $P_\theta \in \Psi^2(\mathbb{R}^n)$ be a complex scaled operator with $\theta = M_1 h\log(1/h)$.

We choose ϵ

$$M_3 h \leq \epsilon \leq M_2 h \log\frac{1}{h}\,, \tag{7.6}$$

where $M_2 > M_1$ and M_3 are large constants to be fixed later.

Let $G \in C^\infty_{\rm c}(T^*X)$ and define

$$\begin{aligned} P_{\epsilon,\theta} &:= e^{-\epsilon G/h} P_\theta e^{\epsilon G/h} \\ &= e^{-\frac{\epsilon}{h}\operatorname{ad}_G} P_\theta \sim \sum_0^\infty \frac{\epsilon^k}{k!}(-\frac{1}{h}\operatorname{ad}_G)^k(P_\theta)\,, \end{aligned}$$

where to simplify notation we write

$$G = G^w(x, hD)\,.$$

We note that the assumption on ϵ and the boundedness of ad_G/h show that the expansion makes sense. The operators $\exp(\epsilon G/h)$ are pseudodifferential in an exotic class $S^{C_2}_\delta$ for any $\delta > 0$ – see Appendix E **More on all this later**

2. Using the same letters for operators and and the corresponding symbols, we see that

$$\begin{aligned} P_{\epsilon,\theta} &= P_\theta - i\epsilon\{P_\theta, G\} + \mathcal{O}(\epsilon^2) \\ &= p_\theta - i\epsilon\{p_\theta, G\} + \mathcal{O}(h+\epsilon^2), \end{aligned}$$

so that

$$\begin{aligned} \mathrm{Re}P_{\epsilon,\theta} &:= (P_{\epsilon,\theta} + P^*_{\epsilon,\theta})/2 \\ &= \mathrm{Re}p_\theta + \epsilon\{\mathrm{Im}p_\theta, G\} + \mathcal{O}(h+\epsilon^2) \\ &= \mathrm{Re}p_\theta + \mathcal{O}(h+\theta\epsilon+\epsilon^2)\,, \end{aligned}$$

and

$$\begin{aligned} \mathrm{Im}P_{\epsilon,\theta} &:= (P_{\epsilon,\theta} - P^*_{\epsilon,\theta})/(2i) \\ &= \mathrm{Im}p_\theta - \epsilon\{\mathrm{Re}p_\theta, G\} + \mathcal{O}(h+\epsilon^2). \end{aligned}$$

3. We now make the following assumption: for a fixed $\delta > 0$

$$|\mathrm{Re}p_\theta - E| < \delta \implies -\mathrm{Im}p_\theta + \epsilon H_p G \geq c_0\epsilon\,. \tag{7.7}$$

We will show how this assumption implies the theorem. It will then remain to construct G such that (7.7) holds.

4. To show how (7.7) implies (7.4) let let $\psi_1, \psi_2 \in C^\infty_{\mathrm{b}}(T^*\mathbb{R})$ be two functions satisfying

$$\psi_1^2 + \psi_2^2 = 1\,, \quad \psi_1|_{|\mathrm{Re}p_\theta|<\delta/2} \equiv 1\,, \quad \mathrm{supp}\psi_1 \subset \{|\mathrm{Re}p_\theta - E| < \delta\}\,.$$

Lemma E.2 gives two selfadjoint operators Ψ_1 and Ψ_2 with principal symbols ψ_1 and ψ_2 respectively, such that

$$\Psi_1^2 + \Psi_1^2 = I + R\,, \quad R = \mathcal{O}(h^\infty)\ :\ H^{-M}(\mathbb{R}) \to H^M\,.$$

We then write

$$P_{\epsilon,\theta} - E = A_{\epsilon,\theta} + iB_{\epsilon,\theta},,$$

where

$$A_{\epsilon,\theta} = \frac{1}{2}(P_{\epsilon,\theta} + P^*_{\epsilon,\theta}) - E$$

and

$$B_{\epsilon,\theta} = \frac{1}{2i}(P_{\epsilon,\theta} - P^*_{\epsilon,\theta})\,.$$

The principal symbol of $B_{\epsilon,\theta}$ is given by

$$\mathrm{Im}p_\theta - \epsilon H_p G$$

and on the essential support of Ψ_1 it is bounded below by $c_0\epsilon \gg h$. The sharp Gårding inequality now implies that for h small enough

$$\begin{aligned}\|(P_{\epsilon,\theta}-E)\Psi_1 u\|\|\Psi_1 u\| &\geq |\langle (P_{\epsilon,\theta}-E)\Psi_1 u, \Psi_1 u\rangle| \\ &\geq |\mathrm{Im}\langle (P_{\epsilon,\theta}-E)\Psi_1 u, \Psi_1 u\rangle| \\ &= -\langle B_{\epsilon,\theta}\Psi_1 u, \Psi_1 u\rangle \\ &\geq \frac{\epsilon}{C}\|\Psi_1 u\|^2\,,\end{aligned}$$

and hence

$$\|(P_{\epsilon,\theta}-E)\Psi_1 u\| \geq \frac{\epsilon}{C}\|\Psi_1 u\|\,.$$

On the support of ψ_2 the operator $A_{\epsilon,\theta}$ is elliptic and by Lemma E.1,

$$\|(P_{\epsilon,\theta}-E)\Psi_2 u\| \geq \frac{1}{C}\|\Psi_2 u\| - \mathcal{O}(h^\infty)\|u\|\,.$$

Applying Lemma E.3 with $t = \epsilon/h \gg 1$ we conclude that

$$\|(P_{\epsilon,\theta}-E)u\| \geq \frac{\epsilon}{C}\|u\|\,.$$

This shows that the conjugated operator has no spectrum in

$$D(E, \epsilon/(2C)) = D(E, Mh\log 1/h)\,.$$

5. We now need to do is to construct G so that (7.7) holds.

Part ii) of Theorem 7.1 implies that

$$K_{[E-2\delta,E+2\delta]} = \emptyset\,,$$

for some $\delta > 0$.

Let us now fix R a large parameter. We will define $G_\rho \in C^\infty_{\rm c}(T^*\mathbb{R})$, a local escape function supported in a neighbourhood of the bicharacteristic segment

$$I_\rho = \{\exp(tH_p)(\rho) :\ t \in [-T,T]\}\,,$$

and which satisfies $H_p G_\rho \geq 1$ on the part of I_ρ lying over

$$K' = \{\rho' \in T^*\mathbb{R}\ :\ |x(\rho')| \leq R\} \tag{7.8}$$

For that, let Γ be a hypersurface through ρ which is transversal to H_p. Then there is a neighbourhood U_ρ of ρ, such that

$$V_\rho = \{\exp(t(U_\rho \cap \Gamma)) :\ t \in (-T-1, T+1)\} \subset \{E-2\delta < p < E+2\delta\}\,,$$

is a neighbourhood of I_ρ. That ne

ighbourhood can be identified with a product,

$$V_\rho \simeq (-T-1, T+1)\times(U_\rho\cap\Gamma)\,,$$

and, in this identification, we will choose T and $0 < \alpha < 1$ so that

$$(((-T-1,-\alpha T)\cup(\alpha T, T+1))\times(U_\rho\cap\Gamma)))\cap K' = \emptyset\,.$$

We now need the following elementary

LEMMA 7.3. *For any $0<\alpha<1/2$ and $T>0$ there exist as function $\chi = \chi_{T,\alpha}\in C^\infty(\mathbb{R};\mathbb{R})$ such that*

$$\chi(t) = \begin{cases} 0 & |t|>T \\ t & |t|<\alpha T \end{cases}\,, \quad \chi'(t)\geq -2\alpha\,.$$

Proof. The piecewise linear function

$$\chi_{\#}(t) = \begin{cases} 0 & |t|>T \\ t & |t|<\alpha T \\ \pm\alpha(T-t)/(1-\alpha) & \alpha T\leq \pm t\leq T \end{cases}$$

satisfies $\chi_\sharp{}' \geq -\alpha/(1-\alpha) > -2\alpha$ wherever the derivative is defined. A regularization of this function gives $\chi_{T,\alpha}$. □

Now let $\varphi_\rho \in C^\infty_{\rm c}(U_\rho\cap\Gamma)$ be identically 1 near ρ, and let χ_T be given by the lemma. Using the product coordinates, we can think of φ_ρ, t, and hence $\chi(t)$, as functions on $T^*\mathbb{R}$. The functions φ_ρ and $\chi_T(t)$ have compact support in V_ρ. Let

$$\psi\in C^\infty_{\rm c}((-\epsilon_0,\epsilon_0))\,, \quad \psi|_{[-\epsilon_0/2,\epsilon_0/2]}\equiv 1\,,$$

and put

$$G_\rho = \chi_T(t)\varphi_\rho\psi(p)\,, \quad G_\rho\in C^\infty_{\rm c}(V_\rho)\,. \tag{7.9}$$

so that

$$H_pG_\rho = \chi_T'\varphi_\rho\psi(p), \tag{7.10}$$

satisfies

$$H_pG_\rho = 1 \text{ on } V_\rho\cap\{|x|<R\} \text{ and } H_pG_\rho\geq -2\alpha \text{ everywhere.}$$

Now let $K\Subset T^*\mathbb{R}$ be the compact set

$$K = \{\rho\in p^{-1}([-\epsilon_0/3,\epsilon_0/3])\ :\ |x(\rho)|\leq R/2\}. \tag{7.11}$$

Since K is compact, applying the previous argument for every $\rho\in K$ gives a U_ρ, and a $U'_\rho\subset U_\rho$ on which $\varphi_\rho = 1$. Since $\{U'_\rho:\ \rho\in K\}$ covers K, the compactness of K shows that we can pass to a finite subcover, $\{U'_{\rho_j}:\ j=1,\dots,N\}$. We let

$$G = \sum_{j=1}^N G_{\rho_j}. \tag{7.12}$$

The construction of G_{ρ_j}'s now shows that by choosing α small enough (depending on the maximal number of support overlaps we obtain

$$\begin{aligned} H_p G(\rho) \geq 1\,, \quad \rho \in p^{-1}((E-2\delta, E+2\delta)) \cap \{|x(\rho)| < R\} \\ H_p G(\rho) \geq -\delta\,, \quad \rho \in T^*\mathbb{R}\,. \end{aligned} \tag{7.13}$$

We now want to choose the scaling so that (7.7) holds with G satisfying (7.13).

For that we choose the complex scaling so that

$$\begin{aligned} -\mathrm{Im} p_\theta(x,\xi) \geq \theta \ \text{ when } |p(x,\xi)| \leq \epsilon_0 \text{ and } |x| \geq R, \\ \mathrm{Im} p_\theta < C_1 \theta \quad \text{when } |p(x,\xi)| \leq \epsilon_0, \end{aligned} \tag{7.14}$$

where R is independent of θ. With $\epsilon = M_2 h \log(1/h)$ we now choose $\theta = M_1 h \log(1/h)$ such that

$$M_1 < M_2/C_1\,, \quad \delta M_2 < M_1\,,$$

where C_1 comes from (7.14) and δ comes from (7.13). Since we can choose δ as small as we want this can certainly be arranged leading to (7.7).

□

7.3. **A lower bound on the resolvent for trapping perturbations.**

In the previous section we have shown that the truncated resolvent satisfies

$$K_E = \emptyset \implies \chi(P - E - i0)^{-1}\chi = \mathcal{O}_{L^2 \to L^2}(1/h)\,.$$

In this section we will consider a *lower bound* on the norm of the resolvent in the case of trapping. In Chapter 8 we will show that this lower bound is achieved in many situations.

THEOREM 7.4 (**Lower bounds on resolvent for trapping perturbations).** *Suppose that $E_0 > 0$ and that $K_{E_0} \neq \emptyset$, and that $\chi \in C_c^\infty(\mathbb{R}^n)$ is equal to 1 near $\pi(K_{E_0})$.*

Then there exists $C_0 = C_0(E_0)$ such that for any $\delta > 0$ there exists $h_0 = h_0(\delta)$ so that

$$\sup_{|E-E_0|<\delta} \|\chi(P - E_0 - i0)^{-1}\chi\|_{L^2 \to L^2} \geq \frac{\log(1/h)}{C_0 h}\,, \tag{7.15}$$

$0 < h < h_0$.

Before giving the proof of Theorem 7.4 we need to present an older result, essentially due to Kato, relating resolvent estimates to *local smoothing* in Schrödinger propagation.

THEOREM 7.5 (Kato's local smoothing). *Let $E_0 > 0$ and let $K(h) \geq 1$ be a function on $(0,1)$.*

Suppose that for $|E - E_0| < \delta$ and $\chi \in L^\infty(\mathbb{R}^n)$ we have

$$\|\chi(P(h) - E - i0)^{-1}\chi\|_{L^2 \to L^2} \leq \frac{K(h)}{h}\,, \tag{7.16}$$

Then for $\varphi \in C_c^\infty((E-\delta, E+\delta); [0,1])$ and $u \in L^2(\mathbb{R}^n)$,

$$\int_{\mathbb{R}} \|\chi\varphi(P)\exp(-itP/h)u\|_{L^2}^2 dt \leq CK(h)\|u\|_{L^2}^2\,. \tag{7.17}$$

for C independent of h.

INTERPRETATION. If the integration in (7.17) takes place over a finite interval in time, $[0,T]$, then the estimate is obvious with $CK(h)$ replaced by T. The localization in space, $\chi(x)$ and in energy, $\varphi(P)$ are also not needed. Hence the point lies in having the integral over $\mathbb{R}$. For that χ for which (7.16) holds is needed. In our presentation we take $\chi \in C_c^\infty(\mathbb{R}^n)$ but finer weights, such as $\langle x\rangle^{-1/2-\epsilon}$ also work – see [VaZw] and references given there.

When $P = -h^2\Delta_g$, where g is a metric, we can rewrite (7.17) as follows

$$\int_{\mathbb{R}} \|\chi\varphi(-h^2\Delta_g)\exp(-it\Delta_g)u\|_{L^2}^2 dt \leq ChK(h)\|u\|_{L^2}\,.$$

If $K(h) = 1$, as is the case in (7.5) under non-trapping assumption, then

$$\int_{\mathbb{R}} \|\chi\varphi(-h^2\Delta_g)(I - \Delta_g)^{1/4}\exp(-it\Delta_g)u\|_{L^2}^2 dt \leq C\|u\|_{L^2}\,.$$

A dyadic decomposition (see for instance [EZ, Section 7.5] for a self-contained presentation in semiclassical spirit) then shows that

$$\int_{\mathbb{R}} \|\chi(1-\psi)(-\Delta_g)\exp(-it\Delta_g)u\|_{H^{\frac12}}^2 dt \leq C\|u\|_{L^2}\,, \tag{7.18}$$

where $\psi \in C_c^\infty(\mathbb{R}; [0,1])$, $\psi \equiv 1$ near 0. To control the term with $\psi(-\Delta_g)$ one needs finer analysis of the bottom of the spectrum of $-\Delta_g$

but a crude bound gives

$$\int_{-T}^{T} \|\chi \exp(-it\Delta_g)u\|^2_{H^{\frac{1}{2}}} dt \leq CT\|u\|_{L^2}\,, \tag{7.19}$$

This is the local smoothing estimate for non-trapping perturbations. In this formulation the smoothing character is clear: we gain 1/2 derivative when localizing in space and averaging in time.

Doi [Doi] showed that any trapping produces a loss in the $H^{1/2}$ regularity. The proof of Theorem 7.4 uses Theorem 7.5 and a semiclassical and quantitative version of his argument to obtain the lower bound $K(h) \geq \log(1/h)/C$.

Proof. 1. It is enough to prove the theorem with the integral in (7.17) over $(0,T)$, with estimates independent of T. This will be done using a TT^* argument.

2. Thus we define

$$A_T : u \longmapsto \mathbb{1}_{[0,T]}\chi\varphi(P)e^{-itP/h}\,,$$
$$A_T : L^2(\mathbb{R}^n) \longrightarrow L^2([0,T]\times\mathbb{R}^n)\,,$$

so that (7.17) is equivalent to

$$\|A_T u\|^2_{L^2_{tx}} \leq CK(h)\|u\|_{L^2_x}\,,$$

or to

$$\|A_T^* f\|^2_{L^2_x} \leq CK(h)\|f\|^2_{L^2_{tx}}\,.$$

This last inequality is equivalent to showing that

$$A_T A_T^* = \mathcal{O}(K(h)) \;:\; L^2([0,T]\times\mathbb{R}^n) \longrightarrow L^2([0,T]\times\mathbb{R}^n)\,, \tag{7.20}$$

with bounds *independent of* T.

3. To obtain (7.20) we start by calculating the adjoint:

$$A_T^* f = \int_0^T e^{isP/h}\varphi(P)\chi f(s)ds\,, \quad f \in L^2_{tx}\,,$$

so that

$$A_T A_T^* f = \mathbb{1}_{[0,T]}(t)\int_{\mathbb{R}} \chi e^{-i(t-s)P/h}\varphi(P)^2\chi\mathbb{1}_{[0,T]}(s)f(s)ds\,.$$

This we can rewrite as $A_T A_T^* f =$

(7.21)

$$\mathbb{1}_{[0,T]}(t)\chi\left(\int_{\mathbb{R}}\sum_{\pm}\mathbb{1}_{\mathbb{R}_\pm}(t-s)\chi e^{-i(t-s)P/h}\varphi(P)^2\right)\chi\mathbb{1}_{[0,T]}(s)f(s)ds$$

$$= \mathbb{1}_{[0,T]}(t)\left(\sum_{\pm}\chi\mathbb{1}_{\mathbb{R}_\pm}(\bullet)\chi e^{-i\bullet P/h}\varphi(P)^2\chi\right) * \left(\mathbb{1}_{[0,T]}(\bullet)f(\bullet)\right)\,,$$

where $*$ denotes the convolution in the t variable.

4. The boundary values of cut-off resolvents on the real axis are and the propagators are related as follows:

$$(P-\lambda\mp i0)^{-1} = \mp\frac{i}{h}\int_{\mathbb{R}}\mathbb{1}_{\mathbb{R}_\pm}(t)e^{-itP/h}e^{it\lambda/h}dt\,,$$

or, in terms of the (unitary) semiclassical Fourier transform, $\mathcal{F}$,

$$(P-\lambda\mp i0)^{-1} = \mp i\sqrt{\frac{2\pi}{h}}\mathcal{F}^*_{t\mapsto\lambda}\left(\exp(-itP/h)\mathbb{1}_{\mathbb{R}_\pm}(t)\right)\,.$$

Returning to (7.21) and using the relation between the Fourier transforms and convolution (paying attention to the factor of $\sqrt{h}$ because of the unitarity of $\mathcal{F}$) we see that

$$A_T A_T^* f = (h/i)\mathbb{1}_{[0,T]}(t)\times$$

$$\mathcal{F}_{\lambda\mapsto t}\left(\left(\sum_{\pm}\pm\chi(P-\lambda\pm i0)^{-1}\varphi(P)^2\chi\right)\mathcal{F}^*_{t\mapsto\lambda}\left(\mathbb{1}_{[0,T]}(t)f(t)\right)\right)\,,$$

We now note that by Stone's formula (B.1) the difference of the resolvent gives the spectral projection of P and consequently we can replace $\varphi(P)$ by $\varphi(\lambda)$ – see (B.2).

5. To conclude the proof we apply Plancherel's formula:

$$\begin{aligned}
&\|A_T A_T^* f\|_{L^2 tx}\\
&\leq h\left\|\left(\sum_{\pm}\pm\chi(P-\lambda\pm i0)^{-1}\chi\right)\varphi(\lambda)^2\mathcal{F}^*_{t\mapsto\lambda}\left(\mathbb{1}_{[0,T]}(t)f(t)\right)\right\|_{L^2_{\lambda x}}\\
&\leq 2h\sup_{\lambda}\|\varphi(\lambda)^2\chi(P-\lambda-i0)^{-1}\chi\|_{L^2_x\to L^2_x}\|f\|_{L^2_{tx}}\\
&\leq 2K(h)\|f\|_{L^2_{tx}}\,,
\end{aligned}$$

Here we used hypothesis (7.16), the assumptions on φ, and the basic fact that the norms of $\chi(P-\lambda \pm i0)^{-1}\chi$ are the same.

This proves (7.20) and consequently (7.17). $\square$

Proof of Theorem 7.4. 1. We will use Theorem 7.5. It shows that if for some nontrivial $u_0 \in L^2(\mathbb{R}^n)$ and

$$\varphi \in C_{\rm c}^\infty((E_0-\delta, E_0+\delta); [0,1])\,, \quad \varphi(E_0) = 1\,,$$

$$\|\chi\varphi(P)\exp(-itP/h)u_0\|^2_{L^2_{tx}} \geq K(h)\|u_0\|_{L^2_x}\,, \tag{7.22}$$

then

$$\sup_{|E-E_0|<\delta} \|\chi(P-E-i0)^{-1}\chi\|_{L^2\to L^2} \geq \frac{K(h)}{Ch}\,.$$

Hence we need to show that for χ satisfying

$$\chi \in C_{\rm c}^\infty(T^*\mathbb{R}^n)\,, \quad \chi \equiv 1 \ \text{ near } \pi(K_{E_0}) \ \,, \tag{7.23}$$

(7.22) holds with

$$K(h) = c\log\frac{1}{h}\,,$$

where c is independent of δ, $0<h<h_0(\delta)$.

2. Functional calculus for pseudodifferential operators (see [D-S, Chapter 8] or [EZ]) shows that

$$\begin{gathered} \varphi(P(h))\chi(x)^2\varphi(P(h)) = a^w(x,hD)\,, \quad a \in \mathcal{S}(T^*\mathbb{R}^n)\,, \\ a(x,\xi) = \chi(x)^2\varphi(p(x,\xi))^2 + \mathcal{O}(h\langle x\rangle^{-\infty}\langle\xi\rangle^{-\infty})\,. \end{gathered} \tag{7.24}$$

We put

$$a_t^w(x,hD) := e^{itP/h}a^w(x,hD)e^{-itP/h}\,.$$

Theorem E.4 shows that for

$$0 < t < \alpha\log\frac{1}{h}\,, \tag{7.25}$$

with α sufficiently small, independent of δ,

$$\begin{gathered} a_t \in S_\gamma(T^*\mathbb{R}^n)\,,\ 0<\gamma<1/2\,, \\ a_t - (\exp tH_p)^*a \in h^{2-3\gamma}S_\gamma(T^*\mathbb{R}^n)\,, \end{gathered} \tag{7.26}$$

with all the symbol estimates uniform for t satisfying (E.2).

3. Hence

(7.27)

$$\begin{aligned}\|\chi\varphi(P)\exp(-itP/h)u_0\|^2_{L^2_{tx}} &= \int_{\mathbb{R}} \langle e^{itP/h}\varphi(P)\chi^2\varphi(P)e^{-itP/h}u_0, u_0\rangle_{L^2_x} dt \\ &\geq \int_0^{\alpha\log(1/h)} \langle a_t^w(x, hD)u_0, u_0\rangle_{L^2_x} dt\,.\end{aligned}$$

It remains to find u_0 such that

$$\langle a_t^w(x, hD)u_0, u_0\rangle \geq \frac{1}{2}\,, \quad \|u_0\|_{L^2(\mathbb{R}^n)} = 1\,, \tag{7.28}$$

uniformly for

$$0 < h < h_0\,, \quad 0 < t < \alpha\log(1/h)\,.$$

4. To find u_0 satisfying (7.28) we choose $(x_0, \xi_0) \in K_{E_0}$ and take for u_0 a coherent state concentrated at (x_0, ξ_0):

$$u_0(x) = (2\pi h)^{-n/4}\exp\left(\frac{i}{h}(\langle x - x_0, \xi_0\rangle + i|x - x_0|^2/2)\right)\,. \tag{7.29}$$

Since K_{E_0} is invariant under the flow

$$\exp(tH_p)(x_0, \xi_0) \in K_{E_0}\,.$$

The assumption (7.23) and the fact that $\varphi(E_0) = 1$ show that

$$(\exp tH_p)^* a(x_0, \xi_0) = 1\,,$$

for all time.

Consequently, (7.26) gives

$$a_t(x_0, \xi_0) = 1 + \mathcal{O}(h^{1/2})\,,$$

uniformly for $0 < t < \alpha \log 1/h$.

The properties of $\langle a_t^w(x, hD)u_0, u_0\rangle$ are implied by the following lemma:

LEMMA 7.6. *Suppose that u_0 is given by* (7.29) *and that $b \in S_\gamma$, $0 < \gamma < 1/2$. Then*

$$\begin{gathered}\langle b^w(x, hD)u_0, u_0\rangle = b(x_0, \xi_0) + e(h)\,, \\ |e(h)| \leq C_n h^{\frac{1}{2}} \max_{|\alpha|=1} \sup_{T^*\mathbb{R}^n} |\partial^\alpha b| \leq C_n(b) h^{1/2-\gamma}\,.\end{gathered} \tag{7.30}$$

Proof. 1. Using the definition of u_0 (7.29) and making a change of variables $x = z + w$, $y = z - w$ we obtain

$$
\begin{aligned}
&\langle b(x, hD)u_0, u_0\rangle \\
&\quad = \frac{1}{(2\pi h)^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} b((x+y)/2, \xi) e^{\frac{i}{h}\langle x-y, \xi\rangle} u_0(y)\overline{u_0}(x)\, dy d\xi dx \\
&\quad = \frac{2^{\frac{3n}{2}}}{(2\pi h)^{\frac{3n}{2}}}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} b(z, \xi) e^{\frac{2i}{h}\langle w, \xi-\xi_0\rangle} e^{-\frac{1}{h}(|z-x_0|^2+|w|^2)}\, dw d\xi dz\,,
\end{aligned}
$$

2. For each fixed z and ξ, the integral in w is

$$
\begin{aligned}
\int_{\mathbb{R}^n} e^{\frac{2i}{h}\langle w, \xi-\xi_0\rangle} e^{-\frac{1}{h}|w|^2}\, dw &= e^{-\frac{1}{h}|\xi-\xi_0|^2}\int_{\mathbb{R}^n} e^{-\frac{1}{h}|w+i(\xi-\xi_0)^2|}\, dw \\
&= 2^{-\frac{n}{2}}(2\pi h)^{\frac{n}{2}} e^{-\frac{1}{h}|\xi-\xi_0|^2}
\end{aligned}
$$

3. Therefore

$$
\begin{aligned}
\langle b(x, hD)u_0, u_0\rangle = \frac{2^{\frac{n}{2}}}{(2\pi h)^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} b(z, \xi) e^{-\frac{1}{h}(|z-x_0|^2+|\xi-\xi_0|^2)}\, dz d\xi \\
b_0(x_0, \xi_0)\frac{2^{\frac{n}{2}}}{(2\pi h)^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{-\frac{1}{h}(|z-x_0|^2+|\xi-\xi_0|^2)}\, dx d\xi + e(h) \\
= C_n(h) b(x_0, \xi_0) + e(h)\,,
\end{aligned}
$$

where $e(h)$ satisfies the estimate of (7.30) and

$$
C_n(h) := \frac{2^n}{(2\pi)^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{i\langle x, \xi\rangle} e^{-\frac{1}{2}(|x|^2+|\xi|^2)}\, dx d\xi.
$$

Taking $b \equiv 1$ and recalling that $\|u_0\|_{L^2} = 1$, we deduce that $C_n(h) = 1$. □

End of proof of Theorem 7.4. 5. We apply the lemma to $b = a_t$ which gives

$$
\langle a_t^w(x, hD)u_0, u_0\rangle \longrightarrow a_t(x_0, \xi_0) = 1\,,
$$

again uniformly in t. Hence (7.28) holds. Using (7.27) we obtain

$$
\|\chi\varphi(P)\exp(-itP/h)u_0\|^2_{L^2_{tx}} \geq \frac{\alpha}{2}\log\frac{1}{h}\,,
$$

which is (7.22) with $K(h) = c\log(1/h)$, as needed for (7.15). □

7.4. **Lower bounds on resonance widths.**

7.5. **From quasimodes to resonances.**

7.6. **Sources and further reading.**

The presentation of classical scattering follows [Ge-S1, Appendix].

Theorem 7.2 was proved by Martinez [Ma-2] following a long tradition of works in scattering theory – see that paper and [TZ] for references. Here we mention the seminal work of Lax and Phillips[LP] and of Vainberg [Vai] providing an abstract frame for obtaining resonance free regions, and the work of Helffer and Sjöstrand [H-S], [Sj-1] on large resonance free regions,

$$K_E = \emptyset \implies \mathrm{Res}(P(h)) \cap D(E, \delta = \emptyset$$

for large classes of operators $P(h)$ with analytic coefficients. The proof given here comes from [S-Z10, Section 4].

Theorem 7.4 was proved by Bony, Burq, and Ramond [B-B-R]. The comment that C is independent of δ was made by J.-F. Bony. For more connections between resolvent estimates and local smoothing for Schrödinger propagators see [Bu],[Dat], and references given there.

8. **Chaotic scattering**

Appendix A. **Notation**

A.1. BASIC NOTATION.

$\mathbb{R}_+ = (0, \infty)$

$\mathbb{R}^n = n$-dimensional Euclidean space

x, y denote typical points in $\mathbb{R}^n$: $x = (x_1, \ldots, x_n),\ y = (y_1, \ldots, y_n)$

$\mathbb{R}^{2n} = \mathbb{R}^n \times \mathbb{R}^n$

$z = (x, \xi), w = (y, \eta)$ denote typical points in $\mathbb{R}^n \times \mathbb{R}^n$:
$z = (x_1, \ldots, x_n, \xi_1, \ldots, \xi_n),\ w = (y_1, \ldots, y_n, \eta_1, \ldots, \eta_n)$

$\mathbb{T}^n = n$-dimensional flat torus $= \mathbb{R}^n/\mathbb{Z}^n$

$\mathbb{C} =$ complex plane

$\mathbb{C}^n =$ n-dimensional complex space

$U \Subset V$ means $\bar{U}$ is a compact subset of V

$\langle x, y\rangle = \sum_{i=1}^n x_i \bar{y}_i =$ inner product on $\mathbb{C}^n$

$|x| = \langle x, x\rangle^{1/2}$

$\langle x\rangle = (1 + |x|^2)^{1/2}$

$\mathbb{M}^{m\times n} = m \times n$-matrices

$\mathbb{S}^n = n \times n$ real symmetric matrices

$A^T =$ transpose of the matrix A

I denotes both the identity matrix and the identity mapping.

$J = \begin{pmatrix} O & I \\ -I & O \end{pmatrix}$

$\sigma(z, w) = \langle Jz, w\rangle =$ symplectic inner product

$\#S =$ cardinality of the set S

$|E| =$ Lebesgue measure of the set $E \subset \mathbb{R}^n$

A.2. **FUNCTIONS, DIFFERENTIATION..**

The support of a function is denoted "supp", and a subscript "c" on a space of functions means those with compact support.

• Partial derivatives:

$$\partial_{x_j} := \frac{\partial}{\partial x_j}, \quad D_{x_j} := \frac{1}{i}\frac{\partial}{\partial x_j}$$

• Multiindex notation: A multiindex is a vector $\alpha = (\alpha_1, \dots, \alpha_n)$, the entries of which are nonnegative integers. The size of α is

$$|\alpha| := \alpha_1 + \cdots + \alpha_n.$$

We then write for $x \in \mathbb{R}^n$:

$$x^\alpha := {x_1}^{\alpha_1} \dots {x_n}^{\alpha_n},$$

where $x = (x_1, \dots, x_n)$.

Also

$$\partial^\alpha := \partial_{x_1}^{\alpha_1} \dots \partial_{x_n}^{\alpha_n}$$

and

$$D^\alpha := \frac{1}{i^{|\alpha|}}\partial_{x_1}^{\alpha_1} \dots \partial_{x_n}^{\alpha_n}.$$

(WARNING: Our use of the symbols "D" and "D^α" differs from that in the PDE textbook [E].)

If $\varphi : \mathbb{R}^n \to \mathbb{R}$, then we write

$$\partial\varphi := (\varphi_{x_1}, \dots, \varphi_{x_n}) = \text{gradient},$$

and

$$\partial^2\varphi := \begin{pmatrix} \varphi_{x_1x_1} & \dots & \varphi_{x_1x_n} \\ & \ddots & \\ \varphi_{x_nx_1} & \dots & \varphi_{x_nx_n} \end{pmatrix} = \text{Hessian matrix}$$

Also

$$D\varphi := \frac{1}{i}\partial\varphi.$$

If φ depends on both the variables $x, y \in \mathbb{R}^n$, we put

$$\partial_x^2\varphi := \begin{pmatrix} \varphi_{x_1x_1} & \dots & \varphi_{x_1x_n} \\ & \ddots & \\ \varphi_{x_nx_1} & \dots & \varphi_{x_nx_n} \end{pmatrix}$$

and

$$\partial^2_{x,y}\varphi := \begin{pmatrix} \varphi_{x_1y_1} & \cdots & \varphi_{x_1y_n} \\ & \ddots & \\ \varphi_{x_ny_1} & \cdots & \varphi_{x_ny_n} \end{pmatrix}.$$

• Jacobians: Let

$$x \mapsto y = y(x)$$

be a diffeomorphism, $y = (y^1, \dots, y^n)$. The Jacobian matrix is

$$\partial y = \partial_x y := \begin{pmatrix} \frac{\partial y^1}{\partial x_1} & \cdots & \frac{\partial y^1}{\partial x_n} \\ & \ddots & \\ \frac{\partial y^n}{\partial x_1} & \cdots & \frac{\partial y^n}{\partial x_n} \end{pmatrix}_{n\times n}.$$

The absolute value of the determinant, $|\det \partial y|$, which is the Jacobian factor in integration is denoted $|\partial y|$.

• Differentiation of determinants: suppose $t \mapsto A(t)$ is a function from $\mathbb{R}$ to invertible $N \times N$ matrices:

$$A \,:\, \mathbb{R} \longrightarrow GL(N, \mathbb{R}),$$

Then

$$\frac{d}{dt}\det A(t) = \operatorname{tr}\left(A(t)^{-1}\frac{dA(t)}{dt}\right)\det A(t), \tag{A.1}$$

and consequently

$$\frac{d}{dt}|\det A(t)|^\alpha = \alpha \operatorname{tr}\left(A(t)^{-1}\frac{dA(t)}{dt}\right)|\det A(t)|^\alpha. \tag{A.2}$$

• Poisson bracket: If $f, g : \mathbb{R}^n \to \mathbb{R}$ are C^1 functions,

$$\{f, g\} := \langle \partial_\xi f, \partial_x g\rangle - \langle \partial_x f, \partial_\xi g\rangle \;= \sum_{j=1}^n \frac{\partial f}{\partial \xi_j}\frac{\partial g}{\partial x_j} - \frac{\partial f}{\partial x_j}\frac{\partial g}{\partial \xi_j}.$$

• The *Schwartz space* is

$$\mathscr{S} = \mathscr{S}(\mathbb{R}^n) := \{\varphi \in C^\infty(\mathbb{R}^n) \mid \sup_{\mathbb{R}^n}|x^\alpha \partial^\beta \varphi| < \infty \text{ for all multiindices } \alpha, \beta\}.$$

We say

$$\varphi_j \to \varphi \quad \text{in } \mathscr{S}$$

provided

$$\sup_{\mathbb{R}^n}|x^\alpha D^\beta(\varphi_j - \varphi)| \to 0$$

for all multiindices α, β

We write $\mathscr{S}' = \mathscr{S}'(\mathbb{R}^n)$ for the space of *tempered distributions*, which is the dual of $\mathscr{S} = \mathscr{S}(\mathbb{R}^n)$. That is, $u \in \mathscr{S}'$ provided $u : \mathscr{S} \to \mathbb{C}$ is linear and $\varphi_j \to \varphi$ in $\mathscr{S}$ implies $u(\varphi_j) \to u(\varphi)$.

We say

$$u_j \to u \quad \text{in } \mathscr{S}'$$

provided

$$u_j(\varphi) \to u(\varphi) \quad \text{for all } \varphi \in \mathscr{S}.$$

A.3. **ELEMENTARY OPERATORS..**

Multiplication operator: $M_\lambda f(x) = \lambda f(x)$

Translation operator: $T_\xi f(x) = f(x - \xi)$

Reflection operator: $Rf(x) := f(-x)$

A.4. **OPERATORS..**

$A^* =$ adjoint of the operator A

$[A, B] = AB - BA =$ commutator of A and B

$\sigma(A) =$ symbol of the pseudodifferential operator A

$\operatorname{spec}(A) =$ spectrum of A.

$\operatorname{tr}(A) =$ trace of A.

We say that a bounded operator B is of *trace class* if

$$\|B\|_{\mathrm{tr}} := \sum \sqrt{\lambda_j} < \infty, \tag{A.3}$$

where the $\lambda_j \geq 0$ are the eigenvalues of the self-adjoint operator B^*B.

• If $A : X \to Y$ is a bounded linear operator, we define the operator norm

$$\|A\| := \sup\{\|Au\|_Y \mid \|u\|_X \leq 1\}.$$

We will often write this norm as

$$\|A\|_{X \to Y}$$

when we want to emphasize the spaces between which A maps.

The space of bounded linear operators from X to Y is denoted $L(X,Y)$; and the space of bounded linear operators from X to itself is denoted $L(X)$.

A.5. ESTIMATES.

• We write
$$f = O(h^\infty) \quad \text{as } h \to 0$$
if for each positive integer N there exists a constant C_N such that
$$|f| \le C_N h^N \quad \text{for all } 0 < h \le 1.$$

• If we want to specify boundedness in the space X, we write
$$f = O_X(h^N)$$
to mean
$$\|f\|_X = O(h^N).$$

• If A is a bounded linear operator between the spaces X, Y, we will often write
$$A = O_{X\to Y}(h^N)$$
to mean
$$\|A\|_{X\to Y} = O(h^N).$$

A.6. PSEUDODIFFERENTIAL OPERATORS..

We cross reference the following terminology from Appendix E. Let M denote a manifold.

• A linear operator $A : C^\infty(M) \to C^\infty(M)$ is called a *pseudodifferential operator* if there exist integers m, k such that for each coordinate patch U_γ, and there exists a symbol $a_\gamma \in S^{m,k}$ such that for any $\varphi, \psi \in C^\infty_{\rm c}(U_\gamma)$
$$\varphi A(\psi u) = \varphi \gamma^* a^{\rm w}_\gamma(x, hD)(\gamma^{-1})^*(\psi u)$$
for each $u \in C^\infty(M)$.

• We write
$$A \in \Psi^{m,k}(M)\,,$$
and also put
$$\Psi(M) := \Psi^{0,0}(M)\,.$$

When $h = 1$, that is we do *not* consider the limit $h \to 0$, we put
$$\Psi^m(M) := \Psi^{m,0}(M).$$

Appendix B. **Functional analysis**

B.1. **Spectral theory. to be re-organized and re-written**

THEOREM B.1 (Stone formula). *The spectral projector is given by boundary values of the resolvent as follows*

$$dE_\lambda(P) = \frac{1}{2\pi i}\left((P-\lambda-i0)^{-1} - (P-\lambda+i0)^{-1}\right) d\lambda\,. \tag{B.1}$$

REMARK. An informal but instructive way of writing (B.1) is

$$\delta(P-\lambda) = \frac{1}{2\pi i}\left((P-\lambda-i0)^{-1} - (P-\lambda+i0)^{-1}\right) \tag{B.2}$$

Let H be a complex Hilbert space with inner product $\langle \cdot, \cdot \rangle$. For a *bounded operator*, $A : H \to H$, we define the adjoint $A^* : H \to H$ using the inner product:

$$\langle Au, v \rangle = \langle u, A^* v \rangle\,.$$

An operator A is *self-adjoint* if $A^* = A$.

THEOREM B.2 (Spectral theorem for bounded operators). *Let A be a bounded self-adjoint operator on H. Then there exist a measure space $(X, \mathcal{M}, \mu)$, a real-valued funtion $f \in L^\infty(X, \mu)$ and a unitary operator $U : HL^2(X,\mu)$ such that*

$$U^* M_f U = A\,,$$

where M_f is the multiplication operator:

$$[M_f u](x) = f(x)u(x)\,, \quad u \in L^2(X, \mu)\,.$$

The same theorem applies to *normal operators*, that is, operators satisfying

$$[A, A^*] = AA^* - A^*A = 0\,.$$

In that case f can be complex valued but otherwise the statement is the same.

DEFINITION. Suppose that A is a bounded operator on H. Then the *spectrum of* A, $\mathrm{Spec}(A) \subset \mathbb{C}$, is defined by

$$\mathrm{Spec}(A) = \complement\{\lambda \in \mathbb{C} \;:\; (A-\lambda)^{-1} : H \to H \text{ exists}\}\,.$$

We say that $\lambda \in \mathrm{Spec}(A)$ is an *eigenvalue* of A, if there exists $u \in H$ such that

$$Au = \lambda u \,. \tag{B.3}$$

Theorem B.2 implies that for a self-ajoint bounded operator A, $\mathrm{Spec}(A) = \overline{\mathrm{image}\,(f)} \Subset \mathbb{R}$.

The following important result concerns spectrum of compact operators: $A : H \to H$ is called compact if the image of $\{u : \|u\| \leq 1\}$ under A is a pre-compact subset of H.

THEOREM B.3 (Spectra of compact operators). *Suppose A is a compact operator on H. Then*

(i) Every $\lambda \in \mathrm{Spec}(A) \setminus \{0\}$ is an eigenvalue of A.

(ii) For all nonzero $\lambda \in \mathrm{Spec}(A) \setminus \{0\}$, there exist N such that

$$\ker(A-\lambda)^N = \ker(A-\lambda)^{N+1} \,.$$

(iii) The eigenvalues can only accumulate at 0.

(iv) $\mathrm{Spec}(A)$ *is countable.*

(v) Every $\lambda \in \mathrm{Spec}(A) \setminus \{0\}$ is a pole of the resolvent operator

$$\lambda \longmapsto (A-\lambda)^{-1} \,.$$

(vi) Suppose in addition that A is self-adjoint. Then there exists an orthonormal set $\{u_k\}_{k\in K} \subset H$, *$K = \{0, 1, 2, \cdots, N\}$ or $K = \mathbb{N}$, such that*

$$Au(x) = \sum_{k\in K} \lambda_k u_k(x) \langle u, u_k \rangle \,, \tag{B.4}$$

where $\lambda_0 \geq \lambda_1 \geq \cdots$ are the non-zero eigenvalues of A.

(vii) Conversely, if (B.4) *holds with $\lambda_j \to 0$ then A is compact.*

One of the most frequently encountered classes of compact operators are inclusions between Hilbert spaces. Here is one which is used in this book:

THEOREM B.4 (Rellich-Kondrachov theorem for unbounded domains). *Suppose that the Hilbert $H \subset L^2(\mathbb{R}^n)$ is defined by the norm*

$$\|u\|_H^2 = \|\langle\xi\rangle^\alpha \hat{u}\|_{L^2(\mathbb{R}^n)}^2 + \|a(x)^{-1}u\|_{L^2(\mathbb{R}^n)}^2 \,,$$
$$\alpha > 0 \,, \quad a(x) > 0 \,, \quad \lim_{|x|\to\infty} a(x) = 0 \,,$$

where $\hat{u}$ is the Fourier transform of u and a is continuous.

Then the inclusion

$$H \hookrightarrow L^2 \quad \textit{is compact} \;.$$

THEOREM B.5 (More on spectrum of self-adjoint operators). *Suppose $A : H \to H$ is a bounded self-adjoint operator.*

(i) *Then $(A - \lambda)^{-1}$ exists and is a bounded linear operator on H for $\lambda \in \mathbb{C} - \operatorname{spec}(A)$, where $\operatorname{spec}(A) \subset \mathbb{R}$ is the spectrum of A.*

(ii) *If $\operatorname{spec}(A) \subset [a, \infty)$, then*

$$\langle Au, u\rangle \geq a\|u\|^2 \quad (u \in A). \tag{B.5}$$

THEOREM B.6 (Maximin and minimax principles). *Suppose that $A : H \to H$ is self-adjoint and semibounded, meaning $A \geq -c_0$. Assume also that $(A + 2c_0)^{-1} : H \to H$ is a compact operator.*

Then the spectrum of A is discrete: $\lambda_1 \leq \lambda_2 \leq \lambda_3 \cdots$; and furthermore

(i)

$$\lambda_j = \max_{\substack{V\subset H \\ \operatorname{codim} V < j}} \min_{\substack{v\in V \\ v\neq 0}} \frac{\langle Av, v\rangle}{\|v\|^2}, \tag{B.6}$$

(ii)

$$\lambda_j = \min_{\substack{V\subset H \\ \dim V \leq j}} \max_{\substack{v\in V \\ v\neq 0}} \frac{\langle Av, v\rangle}{\|v\|^2}. \tag{B.7}$$

In these formulas, V denotes a linear subspace of H.

DEFINITIONS. (i) Let $Q : H \to H$ be a bounded linear operator. We define the *rank* of Q to be the dimension of the range $Q(H)$.

(ii)If A is an operator with real and discrete spectrum, we set

$$N(\lambda) := \#\{\lambda_j \mid \lambda_j \leq \lambda\}$$

to count the number of eigenvalues less than or equal to λ.

THEOREM B.7 (Estimating $\boldsymbol{N(\lambda)}$). *Let A satisfy the assumptions of Theorem B.6.*

(i) *If*

$$\text{(B.8)} \qquad \begin{cases} \text{there exist } \delta > 0 \text{ and a self-adjoint operator } Q, \\ \text{with rank } Q \leq k, \text{ such that} \\ \langle Au, u \rangle \geq (\lambda + \delta)\|u\|^2 - \langle Qu, u \rangle \textit{ for } u \in H, \end{cases}$$

then

$$N(\lambda) \leq k.$$

(ii) *If*

$$\text{(B.9)} \qquad \begin{cases} \text{for each } \delta > 0, \text{ there exists a subspace } V \\ \text{with } \dim V \geq k, \text{ such that} \\ \langle Au, u \rangle \leq (\lambda + \delta)\|u\|^2 \text{ for } u \in V, \end{cases}$$

then

$$N(\lambda) \geq k.$$

Proof. 1. Set W be the orthogonal complement of $Q(H)$, $W := Q(H)^\perp$. Thus codim $W =$ rank Q $\leq k$. Therefore the maximin formula (B.6) implies

$$\begin{aligned} \lambda_{k+1} &= \max_{\substack{V \subset H \\ \text{codim} V < k}} \min_{\substack{v \in V \\ v \neq 0}} \frac{\langle Av, v \rangle}{\|v\|^2} \geq \min_{\substack{v \in W \\ v \neq 0}} \frac{\langle Av, v \rangle}{\|v\|^2} \\ &= \min_{\substack{v \in W \\ v \neq 0}} \left(\lambda + \delta - \frac{\langle Qv, v \rangle}{\|v\|^2} \right) = \lambda + \delta, \end{aligned}$$

since $\langle Qv, v \rangle = 0$ if $v \in Q(H)^\perp$. Hence $\lambda < \lambda + \delta \leq \lambda_{k+1}$, and so

$$N(\lambda) = \max\{j \mid \lambda_j \leq \lambda\} \leq k.$$

This proves assertion (i).

2. The minimax formula (B.7) directly implies that

$$\lambda_k \leq \max_{\substack{v \in V \\ v \neq 0}} \frac{\langle Av, v \rangle}{\|v\|^2} \leq \lambda + \delta.$$

Hence $\lambda_k \leq \lambda + \delta$. This is valid for all $\delta > 0$, and so

$$N(\lambda) = \max\{j \mid \lambda_j \leq \lambda\} \geq k.$$

This is assertion (ii).

□

B.2. **Singular values.**

Let $A : H_1 \to H_2$ be a bounded operator between Hilbert spaces H_1 and H_2. Then $(A^*A)^{\frac{1}{2}} : H_1 \to H_1$ is also a also a bounded operator and we inductivly define

$$\begin{aligned} & s_0(A) = \|A\| \,, \\ & s_{j+1}(A) = \sup\{\lambda \in \operatorname{Spec}(A^*A)^{\frac{1}{2}} \,,\ \lambda < s_j(A)\} \,, \end{aligned} \tag{B.10}$$

with $s_{j+1}(A) = s_{j+p}(A)$ if $s_{j+1}(A)$ lies in the discrete spectrum and its multiplicity is p.

In particular we have,

$$s_0(A) \geq s_1(A) \geq s_2(A) \geq \cdots \,, \quad s_j(A) \in \operatorname{Spec}((A^*A)^{\frac{1}{2}}) \,,$$

When the spectrum of A is discrete, for instance when A is compact operator, then

$$s_0(A) = \|A\| \,, \quad s_j(A) \longrightarrow 0 \,, \quad j \longrightarrow \infty \,.$$

Otherwise the top of the essential spectrum is repeated with infinite multiplicity.

The spectrum of $(AA^*)^{\frac{1}{2}}$ is the same as that of $(A^*A)^{\frac{1}{2}}$ and hence we can define $s_j(A)$'s either ways. They are called the *singular values* of A.

A useful characterization is given in the next theorem.

THEOREM B.8 (Variational characterization of singular values). *Let $A : H_1 \to H_2$ be a bounded operator. Then*

$$s_j(A) = \inf_{\operatorname{rank} K \leq j} \|A - K\|_{H_1 \to H_2} \,. \tag{B.11}$$

The following important result can be easily deduced from Theorem B.8.

THEOREM B.9 (Additive and multiplicative properties of singular values). *1) If $A : H_1 \to H_2$ and $B : H_1 \to H_2$ then*

$$s_{k+\ell}(A+B) \leq s_k(A) + s_\ell(B) \,. \tag{B.12}$$

2) If $A : H_1 \to H_2$ and $B : H_2 \to H_3$ are bounded operators then

$$s_{k+\ell}(AB) \leq s_k(A) s_\ell(B) \,. \tag{B.13}$$

A central inequality between eigenvalues and singular values is due to Weyl:

THEOREM B.10 (Weyl inequalities). *1) Suppose that $A : H_1 \to H_2$ is a bounded operator and $\lambda_j(A)$,*

$$|\lambda_0(A)| \geq |\lambda_1(A)| \geq \cdots |\lambda_N(A)| \,,$$

be its discrete spectrum satisfying

$$\mathrm{Spec}(A) \cap \{\lambda \in \mathbb{C} : |\lambda_0(A)| \geq |\lambda| \geq |\lambda_N(A)|\} = \{\lambda_j(A)\}_{j=1}^N \,,$$

where $\lambda_j(A)$'s are included according to their multiplicities as in (B.10).

If there is a maximal N we put

$$\lambda_k(A) := \inf\{\lambda \in \mathrm{Spec}(A) : |\lambda| > |\lambda_N(A)\} \,, \quad k > N \,.$$

Then for any K we have

$$\prod_{k=0}^{K} (1 + |\lambda_k(A)|) \leq \prod_{k=0}^{K} (1 + s_k(A)) \,. \tag{B.14}$$

2) More generally, let $f : [0, \infty)$, $f(0) = 0$, be a function such that

$$t \longmapsto f(\exp t) \quad \textit{is convex.}$$

Then for any k we have

$$\sum_{k=1}^{K} f(|\lambda_k(A)|) \leq \sum_{k=1}^{K} f(s_k(A)) \,.$$

EXAMPLE. Suppose that (M, g) is compact manifold n dimensional Riemannian manifold and that $-\Delta_M$ is the Laplace-Beltrami operator on M. Then the Weyl law for eigenvalue asymptotics states that

$$|\{\lambda \geq 0 : \lambda^2 \in \mathrm{Spec}(-\Delta_M) \,,\ |\lambda| \leq r\}| = c_n \mathrm{vol}_g\,(M) r^n (1 + o(1)) \,,$$
$$c_n = \mathrm{vol}\,(B_{\mathbb{R}^n}(0,1))/(2\pi)^n \,.$$

If we order the eigenvalues of $-\Delta_M$ as $0 = \lambda_0^2 < \lambda_1^2 \leq \lambda_2^2 \leq \cdots$, it then follows that

$$\lambda_j \geq (c_n \mathrm{vol}_g(M))^{-\frac{1}{n}} j^{\frac{1}{n}} (1 - o(1)) \,, \quad j \to \infty \,, \tag{B.15}$$

and

$$s_j((-\Delta_M - 1)^{-s/2}) \leq C_M j^{-\frac{s}{n}} \,. \tag{B.16}$$

Suppose now that $A : L^2(M) \to H^s(M)$, $s \in \mathbb{N}$. Then

$$\begin{aligned} s_j(A) &\leq s_j((-\Delta_M - 1)^{-s/2}) \|(-\Delta_M - 1)^{s/2} A\|_{L^2 \to L^2} \\ &\leq s_j((-\Delta_M - 1)^{-s/2}) \|A\|_{L^2 \to H^2} \\ &\leq C_A j^{-\frac{s}{n}} \,. \end{aligned} \tag{B.17}$$

B.3. **Trace class operators and determinants.**

When

$$\sum_{j=1}^{\infty} s_j(A) < \infty \,, \tag{B.18}$$

we say that A is of trace class:

$$A \in \mathcal{L}_1 = \mathcal{L}_1(H_1, H_2) \,, \quad \|A\|_{\mathcal{L}_1} := \sum_{j=1}^{\infty} s_j(A) \,.$$

$$|\det(I+A) - \det(I+B)| \leq \|A-B\|_{\mathcal{L}_1} \exp(1 + \|A\|_{\mathcal{L}_1} + \|B\|_{\mathcal{L}_1}) \,. \tag{B.19}$$

B.4. **Regularized determinant.**

Suppose that instead of assuming that A is of trace class we only have the property

$$\sum_{j=1}^{\infty} s_j(A)^p < \infty \,, \text{ for some } p > 1. \tag{B.20}$$

Just as (B.18) defined the trace class of operators, (B.20) defines the p-Schatten class:

$$A \in \mathcal{L}_p = \mathcal{L}_p(H_1, H_2) \,, \quad \|A\|_{\mathcal{L}_1} := \sum_{j=1}^{\infty} s_j(A)^p \,.$$

For $A \in \mathcal{L}_P(H, H)$ (the case of $H_1 = H_2$) we defined the regularized determinant of $I + A$ using the following operator

$$R_p(A) = (I + A) \exp\left(-A + \frac{A^2}{2} - \cdots + \frac{(-A)^{p-1}}{p-1} \right) - I \in \mathcal{L}_1 ,$$

$$\det_p(I + A) := \det(I + R_p(A)) . \tag{B.21}$$

We note that if $A \in \mathcal{L}_q$ for $q < p$, then

$$\det_p(I + A) = \det_q(I + A) \exp\left(\sum_{\ell=q}^{p-1} \frac{\operatorname{tr}(-A)^\ell}{\ell} \right) \tag{B.22}$$

In the case of matrices we know that M^{-1} can expressed using Cramer's rule and hence its norm can be estimated using $|\det M|^{-1}$. There is also an infinite dimensional version of this result:

$$\|(I - K)^{-1}\| \leq \frac{\det(I + (K^*K)^{\frac{p}{2}})}{|\det(I - K^p)\|} , \quad K \in \mathcal{L}_p . \tag{B.23}$$

B.5. **Sources and further reading.** For the moment I suggest [Sj-3, Chapter 5] as a concise reference.

Appendix C. **Fredholm theory**

In this appendix we will describe the role of the Schur complement formula in spectral theory, in particular in analytic Fredholm theory.

C.1. **Grushin problems.**

Linear algebra. The *Schur complement formula* states for two-by-two systems of matrices that if

$$\begin{pmatrix} P & R_- \\ R_+ & R_0 \end{pmatrix}^{-1} = \begin{pmatrix} E & E_+ \\ E_- & E_0 \end{pmatrix},$$

then P is invertible if and only if E_0 is invertible, with

$$P^{-1} = E - E_+ E_0^{-1} E_-, \quad E_0^{-1} = R_0 - R_+ P^{-1} R_-. \tag{C.1}$$

Generalization. We can generalize to problems of the form

$$\begin{pmatrix} P & R_- \\ R_+ & O \end{pmatrix} \begin{pmatrix} u \\ u_- \end{pmatrix} = \begin{pmatrix} v \\ v_+ \end{pmatrix} \tag{C.2}$$

where

$$P : X_1 \to X_2, \quad R_+ : X_1 \to X_+, \quad R_- : X_- \to X_2,$$

for appropriate Banach spaces X_1, X_2, X_+, X_-. We call (C.2) a *Grushin problem.* (In practice, we start with an operator P and build a Grushin problem by choosing $R_\pm$, in which case it is normally sufficient to take $R_0 = 0$.)

If the Grushin problem (C.2) is invertible, we call it *well-posed* and we write its inverse as follows:

$$\begin{pmatrix} u \\ u_- \end{pmatrix} = \begin{pmatrix} E & E_+ \\ E_- & E_0 \end{pmatrix} \begin{pmatrix} v \\ v_+ \end{pmatrix} \tag{C.3}$$

for operators

$$E : X_2 \to X_1, \quad E_0 : X_+ \to X_-, \quad E_+ : X_+ \to X_1, \quad E_- : X_2 \to X_-.$$

LEMMA C.1 (The operators in a Grushin problem). *If* (C.2) *is well-posed, then the operators R_+, E_- are surjective, and the operators E_+, R_- are injective.*

C.2. **Fredholm operators.**

DEFINITIONS. (i) A bounded linear operator $P : X_1 \to X_2$ is called a *Fredholm operator* if the kernel of P,

$$\ker P := \{u \in X_1 \mid Pu = 0\},$$

and the cokernel of P,

$$\operatorname{coker} P := X_2/\overline{PX_1}, \text{ where } PX_1 := \{Pu \mid u \in X_1\},$$

are both finite dimensional.

(ii) The *index* of a Fredholm operator is

$$\operatorname{ind} P := \dim \ker P - \dim \operatorname{coker} P.$$

EXAMPLE. Many important Fredholm operators have the form

$$P = I + K, \tag{C.4}$$

where K a compact operator mapping a Banach space X to itself.

Theorem C.3 below shows that the index does not change under continuous deformations of Fredholm operators (with respect to operator norm topology). Hence for operators of the form (C.4) the index is 0:

$$\operatorname{ind} P = \operatorname{ind}(I + tK) = \operatorname{ind} I = 0 \quad (0 \le t \le 1).$$

□

The connection between Grushin problems and Fredholm operators is this:

THEOREM C.2 (Grushin problem for Fredholm operators). (i) *Suppose that $P : X_1 \to X_2$ is a Fredholm operator.*

Then there exist finite dimensional spaces $X_\pm$ and operators $R_- : X_- \to X_2$, $R_+ : X_1 \to X_+$, for which the Grushin problem (C.2) *is well posed. In particular, $PX_1 \subset X_2$ is closed.*

(ii) *Conversely, suppose that that for some choice of spaces $X_\pm$ and operators $R_\pm$, the Grushin problem* (C.2) *is well posed.*

Then $P : X_1 \to X_2$ is a Fredholm operator if and only if $E_0 : X_+ \to X_-$ is a Fredholm operator; in which case

$$\operatorname{ind} P = \operatorname{ind} E_0. \tag{C.5}$$

Assertion (ii) is particularly useful when the spaces $X_\pm$ are finite dimensional.

Proof. 1. Assume $P : X_1 \to X_2$ is Fredholm. Let $n_+ := \dim\ker P$ and $n_- := \dim\operatorname{coker} P$, and write $X_+ := \mathbb{C}^{n_+}, X_- := \mathbb{C}^{n_-}$. Select then linear operators

$$R_- : X_- \to X_2, \quad R_+ : X_1 \to X_+,$$

of maximal rank such that

$$R_- X_- \cap PX_1 = \{0\}, \quad \ker(R_+|_{\ker P}) = \{0\}.$$

Then the operator

$$\begin{pmatrix} P & R_- \\ R_+ & O \end{pmatrix}$$

has a trivial kernel and is onto. Hence it is invertible, and by the Open Mapping Theorem the inverse is continuous.

In particular, consider P acting on the quotient space $X_1/\ker P$, which is a Banach space since $\ker P$ is closed. We have $n_+ = 0$, and

$$PX_1 = P(X_1/\ker P) = \begin{pmatrix} P & R_- \end{pmatrix} \begin{pmatrix} X_1/\ker P \\ \{0\} \end{pmatrix}$$

is a closed subspace.

2. Conversely, suppose that Grushin problem (C.2) is well-posed. According to Lemma C.1, the operators R_+, E_- are surjective, and the operators E_+, R_- are injective. We take $u_- = 0$. Then

$$\text{(C.6)} \qquad \begin{cases} \text{the equation } Pu = v \text{ is equivalent to} \\ u = Ev + E_+ v_+, \ 0 = E_- v + E_0 v_+. \end{cases}$$

This means that

$$E_- : \operatorname{Im} P \to \operatorname{Im} E_0,$$

and so we can define the induced map

$$E^\# : X_2/\operatorname{Im} P \to X_-/\operatorname{Im} E_0.$$

Since E_- is surjective, so is $E^\#$. Also, $\ker E^\# = \{0\}$. This follows since if $E_- v \in \operatorname{Im} E_0$, we can use (C.6) to deduce that $v \in \operatorname{Im} P$. Hence $E^\#$ is a bijection of the cokernels $X_2 \operatorname{Im} P$ and $X_-/\operatorname{Im} E_0$.

3. Next, we claim that

$$E_+ : \ker E_0 \to \ker P$$

is a bijection. Indeed, if $u \in \ker P$, then $u = E_+ v_+$ and $E_0 v_+ = 0$. Therefore E_+ is onto; and this is all we need check, since E_+ injective.

We conclude that

$$\dim\ker P = \dim\ker E_0, \quad \dim\operatorname{coker} P = \dim\operatorname{coker} E_0.$$

In particular, the indices of P and E_0 are equal. $\square$

THEOREM C.3 (**Invariance of the index under deformations).** *The set of Fredholm operators is open in $L(X_1, X_2)$, and the index is constant in each component.*

Proof. When P is a Fredholm operator, we can use Theorem C.2 to obtain $E_0 : \mathbb{C}^{n_+} \to \mathbb{C}^{n_-}$, with

$$\operatorname{ind} E_0 = n_+ - n_-. \tag{C.7}$$

by the Rank-Nullity Theorem of linear algebra. The Grushin problem remains well-posed (with the same operators $R_\pm$) if P is replaced by P', provided $\|P - P'\| < \epsilon$ for some sufficiently small $\epsilon > 0$. Hence the set of Fredholm operators is open.

Using (C.7) we see that the index of P' is the same as the index of P. Consequently it remains constant in each connected component of the set of Fredholm operators. $\square$

We refer to Hörmander [H2, Sect.19.1] for a comprehensive introduction to Fredholm operators

C.3. **Meromorphic continuation of operators.**

The Grushin problem framework provides an elegant proof of the following standard result:

THEOREM C.4 (**Analytic Fredholm Theory).** *Suppose $\Omega \subset \mathbb{C}$ is a connected open set and $\{A(z)\}_{z\in\Omega}$ is a family of Fredholm operators depending holomorphically on z.*

Then if $A(z_0)^{-1}$ exists at some point $z_0 \in \Omega$, the mapping $z \mapsto A(z)^{-1}$ is a meromorphic family of operators on Ω.

Proof. 1. Fix $z_1 \in \Omega$. We form a Grushin problem for $P = A(z_1)$, as described in the proof of Theorem C.2. The same operators $R_\pm^{z_1}$ also provide a well-posed Grushin problem for $P = A(z)$ for z in some sufficiently small neighborhood $V(z_1)$ of z_1.

According to Theorem C.3

$$\operatorname{ind} A(z) = \operatorname{ind} A(z_0) = 0.$$

Consequently

$$n_+ = n_- = n,$$

and $E_0^{z_1}(z)$ is an $n \times n$ matrix with holomorphic coefficients. The invertibility of $E_0^{z_1}(z)$ is equivalent to the invertibility of $A(z)$.

2. This shows that there exists a locally finite covering $\{\Omega_j\}$ of Ω, and a family of functions f_j, holomorphic in Ω_j, such that if $z \in \Omega_j$, then $A(z)$ is invertible precisely when

$$f_j(z) \neq 0.$$

Indeed, we can define $f_j := \det E_0^z$, where E_0^z exists for $z \in \Omega_j$ by the construction in Step 1. Since Ω is connected and since $A(z_0)$ is invertible for at least one $z_0 \in \Omega$, none of f_j's is identically zero.

So $\det E_0(z)$ a non-trivial holomorphic function in $V(z_1)$; and consequently $E_0(z)^{-1}$ is a meromorphic family of matrices. Applying (C.1), we conclude that

$$A(z)^{-1} = E(z) - E_+(z)E_{-+}(z)^{-1}E_-(z)$$

is a meromorphic family of operators in the neighborhood $V(z_1)$. Since z_1 was arbitrary, $A(z)^{-1}$ is in fact meromorphic in all of Ω. $\square$

C.4. **Gohberg-Sigal theory.** Suppose $M(\lambda)$ is a meromorphic family of Fredhold operators on H. Suppose that $M(\lambda)$ has a pole at $\lambda = \mu$:

$$M(\lambda) = \sum_{k=1}^{K} \frac{M_k}{(\lambda - \mu)^k} + M_0(\lambda)\,, \tag{C.8}$$

where $\lambda \mapsto M_0(\lambda)$ is holomorphic near μ.

We then say that the order of the pole is K and we define the multiplicity of the pole, $m(\mu)$, and the rank at μ:

$$m(\mu) := \operatorname{rank} M_1\,, \quad \operatorname{rank}(\mu) := \dim \sum_{k=1}^{K} \operatorname{Im} M_k\,. \tag{C.9}$$

A root function at μ is a holomorphic function, $\lambda \mapsto \varphi(\lambda) \in H$ such that

$$\lim_{\lambda \to \mu} M(\lambda)\varphi(\lambda) = 0\,, \quad \varphi(\mu) \neq 0\,.$$

The multiplicity of φ, $\operatorname{mult}(\varphi)$, is defined as the order of vanishing of $M(\lambda)\varphi(\lambda)$ at μ. The vector $v = \varphi(\mu)$ is called an eigenvector of $M(\lambda)$ at μ. We define

$$\operatorname{rank}(v) := \max\{\operatorname{mult}(\varphi) \ : \ \varphi(\mu) = v\}\,.$$

We also define

$$\ker(\mu) := \operatorname{span}\{v \in H \ : \ \operatorname{rank} v > 0\}\,.$$

If $\dim \ker(\mu) < \infty$ and for all $v \in \ker(\mu)$, $\mathrm{rank}(v) < \infty$ we define a *canonical system* of eigenvectors $\{v_\ell\}_{1\leq \ell \leq L}$ as follows:

$$\begin{aligned} \mathrm{rank}(v_1) &= \max_{v\in\ker(\mu)} \mathrm{rank}(v)\,, \\ \mathrm{rank}(v_\ell) &= \max_{v\in V} \mathrm{rank}(v)\,, \quad \text{for some } V \subset \ker(\mu), \\ &\qquad\qquad V + \mathrm{span}\{v_1, \cdots, v_{\ell-1}\} = \ker(\mu)\,. \end{aligned}$$

A canonical system of eigenvector is not unique but the ordered set

$$\{r_\ell\}_{1\leq\ell\leq L}\,, \quad r_\ell := \mathrm{rank}(v_\ell)\,,$$

is. We call this set, the set of *partial null multiplicities* and we define the *null multiplicity* of $M(\lambda)$ at μ as

$$N_\mu(M) := \sum_{\ell=1}^{L} r_\ell\,. \tag{C.10}$$

Partial null multiplicities (and hence the null multiplicity as well) are unchanged when $M(\lambda)$ is left or right multiplied by $U(\lambda)$, where $U(\lambda)$ is invertible and holomorphic near μ.

THEOREM C.5. *Suppose that $M(\lambda)$ is a meromorphic family of Fredholm operators with a pole of finite rank at $\lambda = \mu$. If $M_0(\lambda)$ in* (C.8) *has index* 0 *then there exist family of operators $\lambda \mapsto U_j(\lambda)$, $j = 1, 2$, holomorphic and invertible near μ, and operators P_m, $1 \leq m \leq N$, such that, near μ,*

$$\begin{aligned} M(\lambda) &= U_1(\lambda)(P_0 + \sum_{m=1}^{N}(\lambda-\mu)^{k_m}P_m)U_2(\lambda)\,, \quad k_\ell \in \mathbb{Z} \setminus \{0\}\,, \\ P_\ell P_m &= \delta_{\ell m}P_m\,, \quad \mathrm{rank}\, P_\ell = 1\,, \ell > 0\,, \quad \mathrm{rank}(I - P_0) < \infty\,. \end{aligned} \tag{C.11}$$

We see that $M(\lambda)^{-1}$ exists, near μ, as a meromorphic family of operators if and only if $P_0 + \sum_{m=1}^{N} P_m = I$, in which case

$$M(\lambda)^{-1} = U_2(\lambda)^{-1}(P_0 + \sum_{m=1}^{N}(\lambda-\mu)^{-k_m}P_m)U_1(\lambda)^{-1}\,.$$

Invariance of null multiplicities under multiplication by holomorphic invertible operator-valued functions shows that, in the notation of (C.11)

$$N_\mu(M) = \sum_{k_\ell>0} k_\ell\,, \qquad N_\mu(M^{-1}) = \sum_{k_\ell<0} k_\ell\,. \tag{C.12}$$

Theorem C.6 now gives the following result about multiplicities.

THEOREM C.6. *Suppose that $M(\lambda)$ and $M(\lambda)^{-1}$ are meromorphic families of Fredholm operators with poles of finite rank. Then*

$$\frac{1}{2\pi i}\operatorname{tr}\oint_{\mu}\partial_\lambda M(\lambda)M(\lambda)^{-1}d\lambda = N_\mu(M) - N_\mu(M^{-1})\,, \tag{C.13}$$

where the integral is over a positively oriented circle which includes μ and no other pole of $\partial_\lambda M(\lambda)M(\lambda)^{-1}$.

In particular, when $M(\lambda) = I + A(\lambda)$ where $A(\lambda)$ is a meromorphic family of trace class operators then we obtain a formula for the multiplicity of zeros and poles of $\det(I + A(\lambda))$ given by the right hand side of (C.13):

$$\begin{gathered}\frac{1}{2\pi i}\operatorname{tr}\oint_{\mu}\frac{D'(\lambda)}{D(\lambda)}d\lambda = n_+(\mu) - n_-(\mu)\,,\\ D(\lambda) := \det(I + A(\lambda))\,,\quad n_\pm(\mu) := N_\mu((I+A)^{\pm 1})\,.\end{gathered} \tag{C.14}$$

Appendix D. Some complex analysis

Suppose that $f : \mathbb{C} \to \mathbb{C}$ is a holomorphic function. In other words, f is an entire function.

The basic result relating the growth of f to the possible growth of the number of its zeros is the *Jensen formula*:

Suppose that $f(0) \neq 0$. Then

$$\int_0^r \frac{n(t)}{t} dt + \log |f(0)| = \frac{1}{2\pi} \int_0^{2\pi} \log |f(e^{i\theta} r)| d\theta \,, \tag{D.1}$$

where $n(t)$ is the number of zeros of $f(z)$ with $|z| < t$.

From this we get an estimate on the number of zeros of f in a disc of radius r:

$$\begin{aligned} n(r) &\leq \frac{1}{\log 2} \int_r^{2r} \frac{n(t)}{t} dt \\ &\leq \frac{1}{\log 2} \left(\log \max_{|z|=2r} |f(z)| - \log |f(0)| \right) . \end{aligned} \tag{D.2}$$

If $f(0) = 0$ we apply the formula to $f(z)/z^p$ where p is the order of vanishing of f at 0.

We also use the Harnack inequality and the Borel-Carathéodory theorem: for f holomorphic in the closed disc $\overline{D(0,R)}$ and $0 < r < R$ we have

$$\max_{|z| \leq r} |f(z)| \leq \frac{2r}{R-r} \max_{|z| \leq R} \mathrm{Re} f(z) + \frac{R+r}{R-r} |f(0)| \,. \tag{D.3}$$

Another thing which comes up frequently are estimates of canonical products.

We define

$$E_p(z) = (1-z) \exp \left(z + \frac{z^2}{2} + \cdots \frac{z^p}{p} \right) .$$

If a sequence $z_k{}_{k=1}^{\infty}$, $z_k \in \mathbb{C}$, satisfies

$$\sum \frac{1}{|z_n|^{p+1}} < \infty \tag{D.4}$$

then the infinite product

$$P(z) := \prod_{k=1}^{\infty} E_p(z/z_n)$$

conveges and

$$m_P(z) := \frac{1}{2\pi i}\oint_z \frac{P'(w)}{P(w)}dw = \sharp\{k : z_k = z\} .$$

Here the integral is over an "arbitrarily" small positively oriented circle around z.

Using the notation $n(r)$ above we have the following estimate:

$$\max_{|z|\leq r}\log|P(z)| \leq k_p r^p \left(\int_0^r \frac{n(t)}{t^{p+1}}dt + r\int_r^\infty \frac{n(t)}{t^{p+2}}dt\right) . \tag{D.5}$$

In particular, when

$$n(r) \leq Cr^p , \tag{D.6}$$

we have

$$\log|P(z)| \leq C|z|^p . \tag{D.7}$$

A lower bound also holds and here is the case we use. When (D.6) is satisfied then for any $\epsilon > 0$ there existst r_0 such that

$$\log|P(z)| \geq -|z|^{p+\epsilon} , \quad z \notin \bigcup_{m_P(w)>0} D(w, \langle w\rangle^{-p-\epsilon}) , \quad |z| \geq r_0 . \tag{D.8}$$

We say that f is of *exponential type* τ if

$$\limsup_{r\to\infty} \frac{\log\sup_{\lambda\leq r}|f(r)|}{r} = \tau .$$

The type $0 < \tau < \infty$ is called normal.

The indicator function f gives a more precise notion of order:

$$h(\theta) := \frac{\log|f(re^{i\theta})|}{r} .$$

The function h is an indicator function of a convex set $K \subset \mathbb{C}$:

$$h(\theta) = \sup_{z\in K}(\cos\theta\,\mathrm{Re}z + \sin\theta\,\mathrm{Im}z) .$$

The set K is called the indicator diagram of f.

When $h(\theta)$ is a limit along a density one sequence of r's (not just $\limsup$) and the convergence is uniform in θ, the function f is said to have *completely regular growth.* In that case we can describe the distribution of zeros in sectors using the indicator function – see [Le].

Here we quote a specific result which is used in Section 2.4:

THEOREM D.1 (Asymptotics of zeros). *If f is of exponential type in $\mathbb{C}$ and if*

$$\int_{\mathbb{R}} \frac{\log^+ f(x)}{1+x^2} dx < \infty \,, \tag{D.9}$$

then f has completely regular growth and the indicator diagram of f is given by an interval $I_f \subset i\mathbb{R}$.

Writing $m_f(z)$ for the multiplicity of a zero of f we have

$$\lim_{r\to\infty} \frac{1}{r} \sum_{\substack{-\epsilon<\arg z<\epsilon \\ |z|\leq r}} m_f(z) = 0 \,,$$

$$\lim_{r\to\infty} \frac{1}{r} \sum_{\substack{-\epsilon<\arg(-z)<\epsilon \\ |z|\leq r}} m_f(z) = \frac{|I_f|}{2\pi} \,,$$

$$\lim_{r\to\infty} \frac{1}{r} \sum_{\substack{\pi-\epsilon<\arg z\leq\pi+\epsilon \\ |z|\leq r}} m_f(z) = \frac{|I_f|}{2\pi} \,.$$

It is not difficult to check that if f satisfies (D.9) and it has normal type τ then

$$|f(z)| \leq (1+|z|)^N e^{\tau(\mathrm{Im} z)_-} \implies I_f = [-i\tau, 0] \,. \tag{D.10}$$

Appendix E. **Semiclassical microlocal analysis**

We say that $A \in h^{-m}\Psi^k(X)$ is elliptic on $K \Subset T^*X$ if

$$|\sigma(A)|_K| > h^{-m}/C, .$$

This is equivalent to saying

LEMMA E.1. *Suppose $Q \in \Psi^m(X)$ is elliptic at (x_0, ξ_0) , $\|u\|_{L^2} = 1$, and $\mathrm{WF}_h(u)$ is contained in a sufficiently small neighbourhood of $(x_0, \xi_{\ 0})$. Then for h small enough,*

$$\|Qu\|_{L^2} \geq 1/C \,.$$

LEMMA E.2. *Suppose that $\psi_j \in C^\infty_{\mathrm{b}}(T^*X)$, $\psi_1^2 + \psi_2^2 = 1$, $\mathrm{supp}\psi_1 \subset \{(x, \xi) \ : \ |\xi| \leq C\}$. Then, there exist $\Psi_1 \in \Psi^{-\infty}(X)$ and $\Psi_2 \in \Psi^{0,0}(X)$, with principal symbols ψ_1 and ψ_2 respectively, such that*

$$\Psi_1^2 + \Psi_1^2 = I + R \,, \quad R \in h^\infty \Psi^{-\infty}(X) \,, \quad \Psi_j^* = \Psi_j \,.$$

The semiclassical Sobolev spaces, $H^s_h(X)$ are defined by choosing a globally elliptic, self-adjoint operator, $A \in \Psi^1(X)$ (that is an operator satisfying $\sigma(A) \geq \langle \xi \rangle / C$ everywhere) and putting

$$\|u\|_{H^s_h} = \|A^s u\|_{L^2(X)} \,.$$

When $X = \mathbb{R}^n$,

$$\|u\|^2_{H^s_h} \sim \int_{\mathbb{R}^n} \langle h\xi \rangle^{2s} |\mathcal{F}u(\xi)|^2 d\xi \,, \quad \mathcal{F}u(\xi) \stackrel{\mathrm{def}}{=} \int_{\mathbb{R}^n} u(x) e^{-i\langle x, \xi \rangle} dx \,.$$

The following lemma will also be useful:

LEMMA E.3. *Suppose that P_t, $t \in (0, \infty)$, is a family of operators such that*

$$P_t \ : \ H^s_h(X) \ \longrightarrow \ H^{s-m}_h(X) \,,$$

$$\forall \, A \in \Psi^{0,-\infty}(X) \,, \quad \mathrm{ad}_{P_t} A = \mathcal{O}(h) \ : \ L^2(X) \ \longrightarrow \ L^2(X) \,, \quad 0 < h < h_0(t) \,,$$

with the bound depending on A but not on t. Let Ψ_j be as in Lemma E.2 and suppose that

$$\|P_t \Psi_j u\| \geq th\|\Psi_j u\| - \mathcal{O}(h)\|u\| \,, \quad j = 1, 2 \,, \quad u \in C^\infty_{\mathrm{c}}(X) \,.$$

Here the constants in $\mathcal{O}$ are independent of h and t. Then for $t > t_0 \gg 1$ and $0 < h < h_0(t)$,

$$\|P_t u\| \geq th\|u\|/2 \,.$$

In Section 7.3 we use the following important result from [EZ, Chapter 13]:

THEOREM E.4 (Egorov's theorem up to Ehrenfest time). *Suppose that $m \geq 1$ is an order function $P = \mathrm{Op}(p)$, $p \in S(m)$, $p = p_0 + O_{S(m)}(h^2)$, and $p_0 \geq m/C - C$, for some $C > 0$.*

Suppose also that $a \in S$ sastisfies

$$\mathrm{supp} a \subset \{(x, \xi) \; : \; p_0(x, \xi) \leq R\} \,,$$

for some $R > 0$, and define

$$\Gamma_R := \lim_{t \to \infty} \frac{1}{t} \sup_{p_0 \leq R} \log \|\partial \varphi_t\| \,, \quad \varphi_t = \exp t H_{p_0} \,. \tag{E.1}$$

For any $\gamma > \Gamma_R$, $T \geq 0$, and $\delta \in [0, 1/2)$, if

$$0 \leq t \leq T + \frac{\delta}{\gamma} \log \frac{1}{h} \,, \tag{E.2}$$

then

$$\begin{gathered} e^{itP/h} a^{\mathrm{w}}(x, hD) e^{-itP/h} = a_t^{\mathrm{w}}(x, hD) \,, \\ a_t \in S_\delta(m^{-\infty}) \,, \quad a_t - \varphi_t^* a \in h^{2-3\delta} S_\delta(m^{-\infty}) \,, \end{gathered} \tag{E.3}$$

with symbolic estimates uniform in t.

INTERPRETATION. This theorem estimates the length time on which we know that the classical/quantum correspondence remains valid. These correspondece refers to the correspondence between classical and quantum flows:

$$t \mapsto e^{ith} a^w(x, hD) e^{-itP/h}$$

is the quantum evolution of the quantume observable $a^w(x, hD)$.

$$t \mapsto \varphi_t^* a$$

is the classical evolution of the classical observable $a(x, \xi)$

The statement that $a_t - \varphi_t^* a = O(h^{2-3\delta})$ means that the quantum evolution of a^{w} given by the conjugation with $\exp(-itP/h)$ is well approximated by the classical evolution up to the time $\delta/\gamma \log(1/h)$. Till that time we also know that the quantum evolved operator is a quantization of a slightly exotic ($\delta > 0$) classical observable a_t. When we allow $p = p_0 + O_{S(m)}(h)$ then the error becomes $O(h^{1-\delta})$. The assumption $p = p_0 + O_{S(m)}(h^2)$ with p_0 independent of h is natural as the term p_0 is (under further assumptions) invariantly defined up to $O(h^2)$.

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