Math 7386 Problem Set 1

1. Consider the viscid Burgers' equation for a function $u(x,t)$:
$$u_t + u u_x = \nu u_{xx} .$$

(a) Find solutions of this equation by using the transformation
$$\varphi(x,t) = \exp\left(-\frac{1}{2\nu}\int_0^x u(y,t)\,dy\right)$$

(b) Show that there are no solutions of the form
$u(x,t) = \exp(i(kx - \omega t))$

(c) Find all solutions of the traveling form
$$u(x,t) = f(x-ct)$$
defined for all $x \in \mathbb{R}$ and $t \in \mathbb{R}$.
Analyze the behavior of these solutions as $\nu \to 0$.
What is the limit, and what phenomenon is being manifest?

2. For the nonlinear Schrödinger equation
$$i u_t + u_{xx} + \lambda |u|^2 u = 0 ,$$
find solutions of the form $u(x,t) = e^{i(kx-\omega t)} f(x-ct)$,
for which $f(\xi) \to 0$ as $|\xi| \to \infty$ and $f \not\equiv 0$.
These are called solitary waves (or solitons in this case).

3. Find the dispersion relation $\omega = W(k)$ that relates frequency to wavenumber of solutions $u(x,t) = e^{i(kx-\omega t)}$ to the linearized Korteweg-deVries (KdV) equation

$$u_t + c u_x + \varepsilon u_{xxx} = 0$$

and determine the phase velocity and group velocity of wave packets as a function of the wavenumber k.

4. The equations for "polaritons", which are exciton/photon pairs are

$$\begin{cases} i\psi_t = (\omega_1 - i\beta_1)\psi + \gamma\varphi + \lambda|\psi|^2\psi \\ i\varphi_t = \gamma\psi + (\omega_2 - i\beta_2)\varphi - \Delta\varphi \end{cases},$$

where $\psi(x,y,t)$ and $\varphi(x,y,t)$ are complex-scalar-valued functions.

Find the relation between ω and $\langle k_1, k_2 \rangle$ (frequency and wavevector) for solutions of the form

$$\begin{pmatrix} \psi(x,y,t) \\ \varphi(x,y,t) \end{pmatrix} = \begin{pmatrix} V_1 \\ V_2 \end{pmatrix} e^{i(k_1 x + k_2 y - \omega t)}.$$

Because the nonlinearity involves the <u>modulus</u> of ψ, such solutions are possible! But the relation depends on the <u>amplitude</u> of one of the fields. This is characteristic of dispersion relations for nonlinear PDE.