2012 F Math 7386 Problem Set 2

5. Define $\Phi(x) := \begin{cases} -|x|, & n=1 \\ -\frac{1}{2\pi}\log|x|, & n=2 \\ \frac{1}{n(n-2)\alpha(n)}\frac{1}{|x|^{n-2}}, & n\geq 3 \end{cases}$

for $x \in \mathbb{R}^n$, where $\alpha(n)$ is the volume of the unit ball in $\mathbb{R}^n$. Prove that $-\Delta\Phi(x) = \delta_0(x)$ in the rigorous distribution sense.

6. Define $\Phi(x,t) = \begin{cases} \frac{1}{(4\pi t)^{n/2}} e^{-\frac{|x|^2}{4t}}, & t>0 \\ 0, & t\leq 0 \end{cases}$ (the "heat kernel")

$(x\in\mathbb{R}^n)$
$t\in\mathbb{R}$

(a) Prove that $\Phi(x,t)$, as a distribution on $\mathbb{R}^n$ with parameter t, converges to $\delta_0(x)$ as $t \to 0^+$.

(b) Prove that $(\partial_t - \Delta)\Phi(x,t) = \delta_0(x,t)$, where $\delta_0(x,t)$ is the Dirac-delta distribution in $\mathbb{R}^{n+1}$.

7. Prove that $(\Delta + k^2)\frac{e^{ikr}}{4\pi r} = -\delta_0(x)$ in $\mathbb{R}^3$, where $r = (x_1^2 + x_2^2 + x_3^2)^{1/2}$.