Notes on Resonance

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Abstract. These notes develop the mathematics of resonance with the point of view that theory, intuition, and physical interpretation are inseparable. The mathematics involves connections between Fourier analysis, spectral theory, and complex analysis. ... The notes and this abstract are in continual, but not continuous, evolution.

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1 The simplest oscillators in excessive detail

'Tis insightful to examine the simplest models of oscillators in seemingly excessive detail. The insight will be useful when studying more complex models, which are composed of integral superpositions of simple oscillators.

1.1 Complex first-order oscillator

Consider the simple differential equation for a complex scalar function $y(t)$,

$$i\dot{y} = E_0\, y\,, \tag{1.1}$$

in which h is a scalar. The spectrum of the multiplication operator on the right-hand side ($y \mapsto E_0\, y$) is the set $\{E_0\}$, and it is self-adjoint when $E_0 \in \mathbb{R}$. This 1D complex equation is a first-order ODE system for two real functions, the real and imaginary parts of $y(t)$.

Free oscillation. The free oscillation of the system is the one-complex-dimensional family of solutions of the homogeneous equation (1.1),

$$y(t) \;=\; y_0\, e^{-iE_0 t}\,. \tag{1.2}$$

The characteristic (circular) frequency of the oscillator is h.

Response to harmonic forcing. Consider the harmonically forced system

$$i\dot{y} \;=\; E_0\, y + f_0\, e^{-i\omega t}. \tag{1.3}$$

Inserting a harmonic (steady-state) solution $y(t) = y_0\, e^{-i\omega t}$ yields

$$y_0 \;=\; \frac{1}{\omega - E_0}\, f_0\,. \tag{1.4}$$

Notice the appearance of the resolvent $R(\omega) = (E_0 - \omega)^{-1}$ of the multiplication operator. As the forcing frequency approaches that of the free oscillation, the response becomes unbounded. There is no steady-state response to forcing at the characteristic (circular) frequency E_0.

Of course, a free oscillation $y_1\, e^{-iE_0 t}$ may be superimposed upon (added to) this response to obtain another solution of the forced problem. The general solution is the 1-complex-dimensional family of functions

$$y(t) = \frac{f_0}{\omega - E_0}\, e^{-i\omega t} + c\, e^{-iE_0 t}\,, \tag{1.5}$$

in which c is an arbitrary complex number.

The initial-value problem. The solution of the initial-value problem

$$\begin{cases} i\dot{y} \;=\; E_0\, y\,, \\ y(0) \;=\; y_0\,, \end{cases} \tag{1.6}$$

assuming that the system is at rest before time $t = 0$, is

$$y(t) \;=\; y_+(t) \;:=\; \begin{cases} 0 & \text{for } \; t < 0\,, \\ y_0\, e^{-iE_0 t} & \text{for } \; t \geq 0\,. \end{cases} \tag{1.7}$$

The image of the map $[0,\infty) \to \mathbb{C} :: t \mapsto e^{-iE_0 t}$ is a unitary semigroup. The Fourier-Laplace transform of $y(t)$ is defined for $\mathrm{Im}\,(\omega) > 0$,

$$(\mathcal{F}y)(\omega) := \frac{1}{2\pi}\int_{-\infty}^{\infty} y(t)\,e^{i\omega t}\,dt = \frac{1}{2\pi}\int_{0}^{\infty} y(t)\,e^{i\omega t}\,dt = \frac{y_0}{2\pi}\int_{0}^{\infty} e^{-iE_0 t}\,e^{i\omega t}\,dt = \frac{1}{2\pi i}\,\frac{1}{E_0-\omega}\,y_0\,. \tag{1.8}$$

Notice that the Fourier-Laplace transform of the solution of the initial-value problem (the semi-group applied to the initial value y_0) is $1/(2\pi i)$ times the resolvent of the multiplication-by-E_0 operator, applied to the initial value y_0. Here is another way to do this calculation, using the definition of the IVP (1.6) only and not the exponential form of the solution directly:

$$E_0\,(\mathcal{F}y)(\omega) = \frac{1}{2\pi}\int_{0}^{\infty} E_0\,y(t)\,e^{i\omega t}\,dt = \frac{1}{2\pi}\int_{0}^{\infty} i\dot{y}\,e^{i\omega t}\,dt = \frac{1}{2\pi}\int_{0}^{\infty} \omega\,y\,e^{i\omega t}\,dt + \frac{y_0}{2\pi i} = \omega\,(\mathcal{F}y)(\omega) + \frac{y_0}{2\pi i}\,, \tag{1.9}$$

which yields the same result as (1.8).

Notice that the resolvent appears in two ways. It gives (1) the response to a harmonic force and (2) the Fourier-Laplace transform of the semigroup. The connection is made transparent by viewing the solution of the IVP as the response to an impulsive force at $t=0$.

Response to an impulsive force. Another (maybe better) way to formulate the initial-value problem (1.6) is by means of an impulsive force applied at time $t = 0$,

$$i\dot{y} \;=\; E_0\,y + i\,y_0\,\delta(t)\,, \tag{1.10}$$

in which δ is the Dirac delta-function. *By assuming that $y(t) = 0$ for $t < 0$*, one can take the F-L transform of this equation for $\mathrm{Im}\,(\omega) > 0$, and one obtains the simple algebraic equation

$$\omega\,Y(\omega) \;=\; E_0\,Y(\omega) + \frac{i}{2\pi}\,y_0\,, \qquad (\mathrm{Im}\,(\omega) > 0) \tag{1.11}$$

where $Y = \mathcal{F}y$, and one recovers (1.8),

$$Y(\omega) \;=\; \frac{1}{2\pi i}\,\frac{1}{E_0-\omega}\,y_0\,. \qquad (\mathrm{Im}\,(\omega) > 0) \tag{1.12}$$

The inverse Fourier transform gives

$$y(t) \;=\; \frac{y_0}{2\pi i}\int_{-\infty}^{\infty}\frac{1}{E_0-\omega}\,e^{-i\omega t}\,d\omega \;:=\; \frac{y_0}{2\pi i}\lim_{\epsilon\to 0^+}\int_{-\infty}^{\infty}\frac{1}{E_0-(\omega+i\epsilon)}\,e^{-i(\omega+i\epsilon)t}\,d\omega\,. \tag{1.13}$$

The residue calculus (see below) produces again the solution $y(t) = y_0 e^{-iE_0 t}\,\chi_{(0,\infty)}(t)$.

A more physically intuitive point of view is to think of the delta-function as an integral superposition of pure oscillations, so that the forcing function becomes

$$i\,y_0\,\delta(t) \;=\; \frac{i\,y_0}{2\pi}\int_{-\infty}^{\infty} e^{-i\omega t}\,d\omega\,. \tag{1.14}$$

Then $y(t)$ should be a superposition of responses to these harmonic forcings,

$$y(t) = \int_{-\infty}^{\infty} Y(\omega)e^{-i\omega t}d\omega\,. \tag{1.15}$$

From (1.4), the response to the forcing $iy_0e^{-i\omega t}/(2\pi)$ $(\omega \neq h)$ is

$$Y(\omega) \;=\; \frac{1}{2\pi\,i}\,\frac{1}{E_0-\omega}\,y_0\,e^{-i\omega t}\,. \tag{1.16}$$

But if one tries to superimpose these responses (taking the inverse Fourier transform), one obtains, as above,

$$y(t) \;=\; \frac{y_0}{2\pi i}\int_{-\infty}^{\infty} \frac{1}{E_0-\omega}\, e^{-i\omega t}\, d\omega \tag{1.17}$$

which does not converge as written. It can be interpreted as a limit of an integral along $\mathbb{R}+i\epsilon$ as $\epsilon \to 0^+$, as above, which produces the solution $y_+(t) = y_0 e^{-iE_0 t}\,\chi_{(0,\infty)}(t)$ to the IVP with the system at rest for negative time.

If one takes $\epsilon \to 0^-$, one obtains the solution $y_-(t) = -y_0 e^{-iE_0 t}\,\chi_{(-\infty,0)}(t)$, or

$$y(t) \;=\; y_-(t) \;:=\; \begin{cases} -y_0\, e^{-iE_0 t} & \text{for } \; t<0\,, \\ 0 & \text{for } \; t>0\,. \end{cases} \tag{1.18}$$

The average of $y_+(t)$ and $y_-(t)$ is another solution to (1.10),

$$y(t) \;=\; \begin{cases} -\frac{y_0}{2}\, e^{-iE_0 t} & \text{for } \; t<0\,, \\ \;\;\frac{y_0}{2}\, e^{-iE_0 t} & \text{for } \; t>0\,; \end{cases} \tag{1.19}$$

it is obtained by interpreting (1.17) as a principle-value integral

$$y(t) \;=\; \frac{y_0}{\pi i}\; P.V.\!\int_{-\infty}^{\infty} \frac{1}{E_0-\omega}\, e^{-i\omega t}\, d\omega\,. \tag{1.20}$$

Notice that the *difference* of $y_+(t)$ and $y_-(t)$ is a solution of the homogeneous equation $i\dot{y} = hy$ for all time, and gives, as expected, the free oscillation $y(t) = y_0 e^{-iht}$. Of course any two solutions of the impulsively forced system (1.10) differ by a multiple of this free oscillation.

Solution by residue calculus. Let's now compute the solution $y_+(t)$ by residue calculus ($y_-(t)$ is done analogously). Assume $E_0 \in \mathbb{R}$. It is important that $\epsilon > 0$ in the limit in (1.13) because this places the pole of the integrand *below* the contour of integration. For $t < 0$, the integrand decays exponentially in the upper half ω-plane (as $\operatorname{Im}\omega \to \infty$), where it is analytic, and one obtains by contour deformation, $y(t) = 0$. For $t > 0$, the integrand decays exponentially in the lower half ω-plane (as $\operatorname{Im}\omega \to -\infty$), where it has a single simple pole at $\omega = E_0$, and one obtains by contour deformation and the residue calculus, $y(t) = y_0 e^{-iE_0 t}$.

1.2 Real second-order oscillator

A sibling of the complex first-order oscillator is the usual harmonic oscillator,

$$\ddot{y} = -\omega_0^2\, y\,, \tag{1.21}$$

in which $\omega_0 > 0$. It can be complexified by considering y to be complex-valued, but the real and imaginary parts remain decoupled. This is converted into a two-dimensional first-order system by introducing the velocity variable $v = \dot{y}$,

$$\frac{d}{dt}\begin{bmatrix} y \\ v \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\omega_0^2 & 0 \end{bmatrix}\begin{bmatrix} y \\ v \end{bmatrix} \;=\; -iA\begin{bmatrix} y \\ v \end{bmatrix}, \qquad A = i\begin{bmatrix} 0 & 1 \\ -\omega_0^2 & 0 \end{bmatrix}. \tag{1.22}$$

The operator in $\mathbb{C}^2$ represented by the matrix A is self-adjoint with respect to the inner product

$$\left\langle \begin{bmatrix} y_1 \\ v_1 \end{bmatrix}, \begin{bmatrix} y_2 \\ v_2 \end{bmatrix} \right\rangle \;:=\; \omega_0^2\, \bar{y}_1 y_2 + \bar{v}_1 v_2\,. \tag{1.23}$$

The eigenvalues of A are $\pm\omega_0$, and the complementary projections onto the corresponding eigenspaces are

$$P_\pm = \frac{1}{2\pi i}\oint_{C_\pm} (\omega I - A)^{-1} d\omega = \frac{1}{2}\begin{bmatrix} 1 & \mp(i\omega_0)^{-1} \\ \mp i\omega_0 & 1 \end{bmatrix}, \tag{1.24}$$

in which $C_\pm$ is a closed curve in the complex ω-plane encircling $\pm\omega_0$ but not $\mp\omega_0$. Thus $P_\pm$ provide a resolution of the identity operator on $\mathbb{C}^2$ that simultaneously resolves, or diagonalizes, the operator given by A:

$$I = P_+ + P_- \,, \tag{1.25}$$
$$A = \omega_0 P_+ - \omega_0 P_- \,. \tag{1.26}$$

This decomposition is known as the spectral resolution of the identity on $\mathbb{C}^2$ associated with A.

Free oscillations. The free oscillations of this system are

$$u(t) = \begin{bmatrix} y(t) \\ v(t) \end{bmatrix} = e^{-iAt} M \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ -i\omega_0 \end{bmatrix} e^{-i\omega_0 t} + c_2 \begin{bmatrix} 1 \\ i\omega_0 \end{bmatrix} e^{i\omega_0 t} \,, \tag{1.27}$$

in which M is the matrix whose columns are the eigenvectors $[1 \; -i\omega_0]^T$ and $[1 \; i\omega_0]^T$. The characteristic frequency of the system is $\omega_0 > 0$.

Response to harmonic forcing. Consider the harmonically forced oscillator

$$\dot{u} = -iAu + f_0 e^{-i\omega t} \,, \tag{1.28}$$

in which f_0 is a constant vector. A solution of the form $u(t) = u_0 e^{-i\omega t}$ has

$$u_0 = -i(A - \omega I)^{-1} f_0 \,. \tag{1.29}$$

The matrix $(A-\omega)^{-1}$ is the resolvent of A at ω. Notice that it has poles at the spectral values of A, namely $\pm\omega_0$.

Starting back at the forced second-order equation

$$\ddot{y} = -\omega_0^2 y + f_0 e^{-i\omega t} \,, \tag{1.30}$$

in which f_0 is a scalar, one finds that the vector $[0 \; f_0]^T$ belongs in the place of the vector f_0 in (1.28).

The initial-value problem, or response to an impulsive force. Consider the system to be at rest until time $t = 0$, when it is instantaneously forced into a new state $u_0 \in \mathbb{C}^2$,

$$\dot{u} = -iAu + u_0 \, \delta(t) \,. \tag{1.31}$$

In the Fourier-Laplace variable ω ($\operatorname{Im}\omega > 0$), with $\mathcal{F}u = U$, this becomes

$$-i\omega \, U = -i \, A \, U + \frac{u_0}{2\pi} \,,$$

with solution

$$U(\omega) = \frac{1}{2\pi i} \, (A - \omega I)^{-1} u_0 \,.$$

Thus the solution $u(t)$ is

$$u(t) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} (A - \omega I)^{-1} u_0 \, e^{-i\omega t} \, d\omega := \frac{1}{2\pi i} \lim_{\epsilon \to 0^+} \int_{-\infty}^{\infty} (A - (\omega + i\epsilon) I)^{-1} u_0 \, e^{-i(\omega + i\epsilon)t} \, d\omega \,.$$

The resolvent $(A-\omega)^{-1}$ is meromorphic with simples poles at $\pm\omega_0$. Its spectral representation is (see 1.26):

$$(A - \omega I)^{-1} = \frac{1}{\omega_0 - \omega} P_+ - \frac{1}{\omega_0 + \omega} P_- \,.$$

The contributions from the poles at $\pm\omega_0$ are $P_+ u_0 \, e^{-i\omega_0 t}$ and $P_- u_0 \, e^{i\omega_0 t}$, and thus the solution is

$$u(t) = P_+ u_0 \, e^{-i\omega_0 t} + P_- u_0 \, e^{i\omega_0 t} \,. \tag{1.32}$$

This expected solution is also obtained simply by decomposing u_0 into its eigenvector components and letting them evolve at exponential rates given by the eigenvalues $-i\omega_0$ and $i\omega_0$ of $-iA$.

1.3 The free Schrödinger equation on the line

The 1-dimensional Schrödinger equation for a function $u(x,t)$ is

$$i\partial_t u = -\mu\partial_{xx}u + V(x)u\,, \tag{1.33}$$

with $\mu > 0$. The free Schrödinger equation has no potential, that is, $V = 0$. The operator $-\mu\partial_{xx}$ is a positive closed operator in $L^2(\mathbb{R})$, whose domain is the Sobolev space $H^2(\mathbb{R})$.

Free oscillations. All solutions of $i\partial_t u = -\mu\partial_{xx}u$ of the form $u(x,t) = \psi(x)e^{-i\omega t}$ $(\omega \neq 0)$ are

$$u(x,t) \;=\; c_1 e^{i(kx-\omega t)} + c_2 e^{-i(kx+\omega t)}\,, \qquad k = \sqrt{\omega/\mu}\,. \tag{1.34}$$

The branch cut for the square root is taken to be the positive real axis, with $\mathrm{Im}\sqrt{\omega} > 0$ if ω is not on the branch cut, and $\sqrt{\omega} \geq 0$ otherwise. When $\omega > 0$, $e^{i(kx-\omega t)}$ is a *rightward traveling* wave, and $e^{-i(kx+\omega t)}$ is a *leftward traveling* wave. (The relation $\omega = \mu k^2$ demonstrates the characteristic *quadratic dispersion* of the Schrödinger equation.)

A general solution of the unforced equation $i\partial_t u = -\mu\partial_{xx}u$ is a linear integral superposition of harmonic oscillations with positive frequency,

$$u(x,t) \;=\; \int_0^\infty \Big(c_1(\omega)\, e^{i(kx-\omega t)} + c_2(\omega)\, e^{-i(kx+\omega t)} \Big)\, d\omega\,, \qquad k = \sqrt{\omega/\mu}\,. \tag{1.35}$$

As an integral in the wavenumber k, one can write

$$u(x,t) \;=\; \int_{-\infty}^{\infty} c(k)\, e^{i(kx-\mu k^2 t)}\, dk\,, \tag{1.36}$$

in which $\pm k$ correspond to left- and right-traveling waves at the same frequency.

The free Schrödinger equation is actually an integral superposition of decoupled harmonic oscillators over all positive frequencies, with two simple oscillators per frequency. The spectral theory shows that this is true for the more general equation $i\partial_t u = Au$, where A is a self-adjoint operator in a Hilbert space.

Harmonically forced oscillations. Consider the harmonically forced system

$$i\partial_t u = -\mu\partial_{xx}u + f(x)e^{-i\omega t}\,. \tag{1.37}$$

Setting $u(x,t) = \psi(x)e^{-i\omega t}$, one obtains the forced Helmholtz equation

$$(\mu\,\partial_{xx} + \omega)\,\psi \;=\; f(x)\,. \tag{1.38}$$

When the forcing is concentrated at $x = 0$, that is f is the Dirac delta-function δ, we call the solution $G(x;\omega)$. It satisfies the distributional equation

$$(\mu\,\partial_{xx} + \omega)\,\psi \;=\; \delta(x)\,. \tag{1.39}$$

If $\omega \ngeq 0$, the L^2 solution is

$$G(x;\omega) \;=\; \frac{1}{2i\sqrt{\mu}\,\sqrt{\omega}}\,\exp\left(i\sqrt{\tfrac{\omega}{\mu}}\,|x|\right)\,, \tag{1.40}$$

in which $\mathrm{Im}\sqrt{\omega} > 0$. [Exercise: Prove this in the rigorous distributional sense.] A forcing concentrated at $x - y$ results in G shifted by y,

$$(\mu\,\partial_{xx} + \omega)\, G(x-y;\,\omega) \;=\; \delta(x-y)\,. \tag{1.41}$$

The force $f(x)$ in the steady-state forced Schrödinger equation (1.38) can be considered to be an integral superposition of shifted delta-functions, $f(x) = \int \delta(x-y)\, f(y)\, dy$, and the principle of linear superposition yields the solution

$$\psi(x,t) \;=\; \int G(x-y,\,\omega) f(y)\, dy \;=\; \frac{1}{2i\sqrt{\mu}\,\sqrt{\omega}} \int f(y)\, \exp\left(i\sqrt{\tfrac{\omega}{\mu}}\,|x-y|\right) dy\,. \tag{1.42}$$

This solution is valid for any $f \in L^2(\mathbb{R})$ and thus represents the resolvent of the operator $L := -\mu\partial_{xx}$,

$$(\omega I - L)^{-1} f\,(x) \;=\; \int G(x-y,\,\omega) f(y)\,dy\,. \tag{1.43}$$

If $\omega > 0$, then $G(\cdot\,;\omega)$ is not in L^2. The solution (1.42) still makes sense for a restricted class of forcing functions, such as those in $L^1 \cap L^2$. This will be the key to an analytic extension of the resolvent across the real ω axis.

The initial-value problem, or the impulsively forced equation. Let the system be at rest for $t < 0$, and let it be instantaneously forced to an initial value of $u_0 \in L^2$ at $t = 0$, and then allowed to oscillate freely:

$$i\partial_t u = -\mu\partial_{xx} u + i\, u_0(x)\delta(t)\,. \tag{1.44}$$

Taking the Fourier-Laplace transform yields the same equation as (1.38),

$$(\mu\,\partial_{xx} + \omega)\, U(x;\,\omega) \;=\; i\frac{u_0(x)}{2\pi}\,, \tag{1.45}$$

or

$$U(x;\,\omega) \;=\; \frac{1}{2\pi i}(L - \omega I)^{-1} u_0\,(x)\,. \tag{1.46}$$

The solution $u(x,t)$ is

$$\begin{aligned} u(x,t) \;&=\; \frac{1}{2\pi i}\int (L-\omega I)^{-1} u_0(x) e^{-i\omega t}\,d\omega \;=\; \frac{i}{2\pi}\int u_0(y)\int G(x-y,\,\omega)\,e^{-i\omega t}\,d\omega\,dy \\ &=\; \frac{1}{4\pi\sqrt{\mu}}\int u_0(y)\int \frac{1}{\sqrt{\omega}}\exp\left(i\sqrt{\tfrac{\omega}{\mu}}\,|x-y|\right)\,e^{-i\omega t}\,d\omega\,dy \;=\; \int u_0(y) K(x-y,\;t)\,dy\,. \end{aligned} \tag{1.47}$$

The inner integral comes out to the integral kernel K for the unitary semigroup of the anti-self-adjoint operator $i\mu\partial_{xx}$. It describes how an initial condition concentrated at $x\!=\!y$ spreads out under the evolution of the free Schrödinger equation. Here, it is given as a superposition of free oscillations.

To find a closed-form expression of this integral kernel, we go through the Fourier transform in the x-variable.

$$u_0(x) = \int \hat{u}_0(k) e^{ikx}\,dk\,, \tag{1.48}$$

$$\hat{u}_0(k) = \frac{1}{2\pi}\int u_0(x) e^{-ikx}\,dx\,. \tag{1.49}$$

Since ∂_x goes over to multiplication by ik under the Fourier transform, one has, for $f \in \mathcal{D}(L)$ (recall $L = -\mu\partial_{xx}$),

$$(Lf)\hat{}\,(k) = \mu k^2 \hat{f}(k) \tag{1.50}$$

and thus

$$\hat{U}(k,\omega) = \frac{1}{2\pi i}\left[(L-\omega I)^{-1} u_0\right]\hat{}\,(k) \;=\; \frac{1}{2\pi i}\,\frac{1}{\mu k^2 - \omega}\,\hat{u}_0(k)\,. \tag{1.51}$$

Returning to the t-variable, one has

$$\hat{u}(k,t) = \frac{-1}{2\pi i}\int \frac{\hat{u}_0(k)}{\omega - \mu k^2}\,e^{-i\omega t}\,d\omega\,, \tag{1.52}$$

Then, returning to the x-variable, one obtains

$$u(x,t) \;=\; \frac{-1}{2\pi i}\iint \frac{\hat{u}_0(k)}{\omega-\mu k^2}\, e^{i(kx-\omega t)}\,d\omega\,dk \;=\; \chi_{(0,\infty)}(t)\int \hat{u}_0(k)e^{i(kx-\mu k^2 t)}\,dk \qquad \text{(residue calculus)}$$

$$= \chi_{(0,\infty)}(t)\frac{1}{2\pi}\iint u_0(y)\, e^{i\left(k(x-y)-\mu k^2 t\right)}\,dy\,dk \qquad \text{(formula for } \hat{u}_0\text{)}$$

$$= \chi_{(0,\infty)}(t)\lim_{\epsilon\to 0}\frac{1}{2\pi}\int u_0(y)\int e^{i\left(k(x-y)-\mu k^2 t\right)-\epsilon k^2}\,dk\,dy \qquad \text{(dominated convergence and Fubini's)}$$

$$= \chi_{(0,\infty)}(t)\frac{1}{\sqrt{2\pi i\mu t}}\int u_0(y)\, e^{i\frac{(x-y)^2}{4\mu t}}\,dy\,. \qquad \text{(contour integration)} \quad (1.53)$$

Thus we have computed K,

$$K(x-y,\,t) \;=\; \chi_{(0,\infty)}(t)\frac{1}{\sqrt{2\pi i\mu t}}\,\exp\left(i\frac{(x-y)^2}{4\mu t}\right)\,. \tag{1.54}$$

Observe the similarity in form to the heat kernel. Because of the i in the exponential, it is actually a completely different thing from the heat kernel.

Spectral resolution of the identity for $-\mu\partial_{xx}$**.** This stuff may not be necessary for the subsequent development of resonance, but it places it in the context of spectral theory for self-adjoint operators in Hilbert space. Since ∂_x is diagonalized (it becomes a multiplication operator) under the Fourier transform, the associated spectral resolution of the identity $\{E_k\}_{k\in\mathbb{R}}$ is obtained by projecting onto Fourier modes, that is,

$$(E_k f)\hat{}= \hat{f}\chi_{(-\infty,k)}, \tag{1.55}$$

or, concretely, $E_k f$ is a truncated integral of the Fourier modes of f:

$$(E_k f)(x) = \int_{-\infty}^{k}\hat{f}(\ell)\,e^{i\ell x}\,d\ell\,. \tag{1.56}$$

The family of operators E_k has the following properties:

- E_k is an orthogonal projection;
- $E_k f \to 0$ as $k\to-\infty$;
- $E_k f \to f$ as $k\to\infty$;
- $E_k E_\ell = E_{\min\{k,\ell\}}$, that is, $\{E_k\}$ is an increasing family.

Thus, one has

$$\int_{\mathbb{R}} dE_k f = \lim_{L\to\infty}\int_{-L}^{L} dE_k f = f\,. \tag{1.57}$$

For an interval $\Delta=(k_1,k_2)$, one has

$$E_\Delta f := E_{k_2}f - E_{k_1}f = \int_\Delta dE_k f = \int_{k_1}^{k_2}\hat{f}(k)\,e^{ik(\cdot)}\,dk\,. \tag{1.58}$$

Using the fact that $(f')\check{}(k) = ik\hat{f}(k)$ for $f\in H^1(\mathbb{R})$, one obtains an integral representation of f' in terms of the Fourier resolution of the identity:

$$f' = \lim_{L\to\infty}\int_{-L}^{L} ik\hat{f}(k)e^{ik(\cdot)}dk = \lim_{\substack{|\mathcal{P}|\to 0\\ L\to\infty}}\sum_{\Delta\in\mathcal{P}} ik_\Delta\int_\Delta \hat{f}(k)e^{ik(\cdot)}dk = \lim_{\substack{|\mathcal{P}|\to 0\\ L\to\infty}}\sum_{\Delta\in\mathcal{P}} ik_\Delta E_\Delta f = \int_{\mathbb{R}} ik\,dE_k f\,. \tag{1.59}$$

The latter integral is a *Riemann-Stieljes integral with respect to the spectral family* $\{E_k\}$. In a similar manner, one obtains the standard spectral representation for $L=-\mu\partial_{xx}$:

$$Lf = -\mu f'' = \int \mu k^2 dE_k f = \int_0^\infty \omega\, d\left(E_{-\sqrt{\omega/\mu}} - E_{\sqrt{\omega/\mu}}\right) f = \int_0^\infty \omega\, d\tilde{E}_\omega f\,, \tag{1.60}$$

in which the spectral family $\{\tilde{E}_k\}$ associated with L is defined by $\tilde{E}_\omega = E_{-\sqrt{\omega/\mu}} - E_{\sqrt{\omega/\mu}}$, or

$$\tilde{E}_\omega f = \begin{cases} 0 & (\omega < 0), \\ \left(\hat{f}\, \chi_{(-\sqrt{\omega/\mu}, \sqrt{\omega/\mu})}\right)\check{} & (\omega \geq 0). \end{cases} \tag{1.61}$$

1.4 The wave equation on the line

The 1D wave equation is

$$\partial_{tt} u \;=\; c^2\, \partial_{xx} u =: -Lu\,. \tag{1.62}$$

The operator $L = -c^2\, \partial_{xx}$ with domain $H^2(\mathbb{R}) \subset L^2(\mathbb{R})$ is self-adjoint and positive. This equation is put into first-order-in-time form by setting $v = \partial_t u$,

$$\partial_t \begin{bmatrix} u \\ v \end{bmatrix} \;=\; \begin{bmatrix} 0 & I \\ -L & 0 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} \;=\; -iA \begin{bmatrix} u \\ v \end{bmatrix}. \tag{1.63}$$

The operator A is self-adjoint in the Hilbert space $H^1(\mathbb{R}) \oplus H^0(\mathbb{R})$ (where $H^0 = L^2$), where its domain is

$$\mathcal{D}(A) \;=\; H^2(\mathbb{R}) \oplus H^1(\mathbb{R}) \;\subset\; H^1(\mathbb{R}) \oplus H^0(\mathbb{R})\,. \tag{1.64}$$

This means that $\mathcal{D}(A) = \mathcal{D}(A^*)$ and that, with respect to the inner product

$$\left\langle \begin{bmatrix} u_1 \\ v_1 \end{bmatrix}, \begin{bmatrix} u_2 \\ v_2 \end{bmatrix} \right\rangle \;=\; \int_{\mathbb{R}} \left(\bar{u}_1' u_2' + \bar{v}_1 v_2\right), \tag{1.65}$$

one has, for all $[u_1\; v_1]^T$ and $[u_2\; v_2]^T$ in $\mathcal{D}(A)$,

$$\left\langle A \begin{bmatrix} u_1 \\ v_1 \end{bmatrix}, \begin{bmatrix} u_2 \\ v_2 \end{bmatrix} \right\rangle \;=\; \left\langle \begin{bmatrix} u_1 \\ v_1 \end{bmatrix}, A \begin{bmatrix} u_2 \\ v_2 \end{bmatrix} \right\rangle. \tag{1.66}$$

[Exercise: Prove this.]

Response to harmonic forcing. The harmonically forced wave equation is

$$\partial_{tt} u \;=\; c^2\, \partial_{xx} u + f(x) e^{-i\omega t}\,. \tag{1.67}$$

In first-order form, it is

$$\partial_t \begin{bmatrix} u \\ v \end{bmatrix} \;=\; \begin{bmatrix} 0 & I \\ -L & 0 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} + \begin{bmatrix} 0 \\ f(x)\, e^{-i\omega t} \end{bmatrix}. \tag{1.68}$$

Put $[u(x,t),\, v(x,t)] = [u(x),\, v(x)] e^{-i\omega t}$ to obtain $(L - \omega^2) u = f$, or the pair $v(x) = -i\omega\, u(x)$ and

$$\left(c^2 \partial_{xx} + \omega^2\right) u \;=\; -f\,. \tag{1.69}$$

With $f \in L^2$, the solution is

$$u \;=\; -(c^2 \partial_{xx} + \omega^2)^{-1} f \;=\; (L - \omega^2 I)^{-1} f\,. \tag{1.70}$$

For $\operatorname{Im}\omega > 0$, the resolvent of $L = -c^2 \partial_{xx}$ at ω^2 is given by (see 1.38–1.42)

$$\left[(L - \omega^2 I)^{-1} f\right](x) \;=\; \frac{i}{2c\,\omega} \int f(y) \exp\left(i \frac{\omega}{c} |x - y|\right) dy\,. \tag{1.71}$$

The initial-value problem. The Fourier transform converts A into an operator of matrix multiplication,

$$\begin{aligned} \mathcal{F}\left\{ A \begin{bmatrix} u \\ v \end{bmatrix} \right\}(k) \;&=\; \mathcal{F}\left\{ i \begin{bmatrix} 0 & I \\ c^2 \partial_{xx} & 0 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} \right\}(k) \\ &=\; i \begin{bmatrix} 0 & I \\ -(ck)^2 & 0 \end{bmatrix} \begin{bmatrix} \hat{u}(k) \\ \hat{v}(k) \end{bmatrix} \;=\; \left[ck\, P_+(ck) - ck\, P_-(ck)\right] \begin{bmatrix} \hat{u}(k) \\ \hat{v}(k) \end{bmatrix}. \end{aligned} \tag{1.72}$$

The projections $P_\pm$ are defined above in (1.24), with ck in place of ω_0. The Fourier transform of the resolvent of A is

$$\mathcal{F}\left\{(A-\omega I)^{-1}\begin{bmatrix}u\\v\end{bmatrix}\right\}(k) \;=\; \left[\frac{1}{ck-\omega}\,P_+(ck)-\frac{1}{ck+\omega}\,P_-(ck)\right]\begin{bmatrix}\hat{u}(k)\\\hat{v}(k)\end{bmatrix}. \tag{1.73}$$

The solution of the initial-value problem

$$\partial_t\begin{bmatrix}u\\v\end{bmatrix} \;=\; -iA\begin{bmatrix}u\\v\end{bmatrix}, \qquad \begin{bmatrix}u(0)\\v(0)\end{bmatrix} \;=\; \begin{bmatrix}u_0\\v_0\end{bmatrix}, \tag{1.74}$$

is then

$$\begin{aligned} u(x,t) \;&=\; \frac{1}{2\pi i}\int(A-\omega I)^{-1}\begin{bmatrix}u_0\\v_0\end{bmatrix}e^{-i\omega t}\,d\omega \;=\; \frac{1}{2\pi i}\iint\mathcal{F}\left\{(A-\omega I)^{-1}\begin{bmatrix}u_0\\v_0\end{bmatrix}\right\}(k)\,e^{i(kx-\omega t)}\,dk\,d\omega\\ &=\; \frac{1}{2\pi i}\iint\left[\frac{1}{ck-\omega}\,P_+(ck)-\frac{1}{ck+\omega}\,P_-(ck)\right]\begin{bmatrix}\hat{u}_0(k)\\\hat{v}_0(k)\end{bmatrix}e^{i(kx-\omega t)}\,d\omega\,dk\\ &=\; \chi_{(0,\infty)}(t)\int_{\mathbb{R}}\left[P_+(ck)\begin{bmatrix}\hat{u}_0(k)\\\hat{v}_0(k)\end{bmatrix}e^{ik(x-ct)}-P_-(ck)\begin{bmatrix}\hat{u}_0(k)\\\hat{v}_0(k)\end{bmatrix}e^{ik(x+ct)}\right]dk\\ &=\; \chi_{(0,\infty)}(t)\int_0^\infty\ldots\,dk\,. \end{aligned} \tag{1.75}$$

This expression represents the solution of the wave equation as a superposition of forward and backward traveling oscillatory waves, all traveling at the same speed c. But what the solution looks like in terms of the initial conditions is not at all transparent.

Let's look at deriving the solution by computing the resolvent directly in the spatial domain. We must solve

$$(A-\omega I)\begin{bmatrix}U\\V\end{bmatrix} = \begin{bmatrix}g\\f\end{bmatrix}, \tag{1.76}$$

which is equivalent to the system

$$c^2U''+\omega^2U = -(\omega g+if)\,, \tag{1.77}$$

$$V = -i(g+\omega U)\,. \tag{1.78}$$

The solution for U is

$$U(x,\omega) = \frac{-1}{2ic}\int\left(g(y)+\frac{if(y)}{\omega}\right)e^{i\frac{\omega}{c}|x-y|}dy \tag{1.79}$$

Now taking the inverse F.L. transform of $U(x,\omega)$ with $g=u_0$ and $f=v_0$, one obtains

$$\begin{aligned} u(x,t) \;&=\; \frac{1}{2\pi i}\int(A-\omega I)^{-1}\begin{bmatrix}u_0\\v_0\end{bmatrix}e^{-i\omega t}\,d\omega = \frac{1}{4\pi c}\iint\left(u_0(y)+\frac{iv_0(y)}{\omega}\right)e^{i\frac{\omega}{c}(|x-y|-ct)}dy\,d\omega\\ &=\; \frac{1}{2}\left(u_0(x-ct)+u_0(x+ct)\right)+\frac{1}{2c}\int_{x-ct}^{x+ct}v_0(y)\,dy\,. \end{aligned} \tag{1.80}$$

[Exercise: Prove the last equality].

Notice that the result is easily obtainable by d'Alembert's method. But when a defect is placed on the string, as in the Lamb model below, doing these integrals becomes a powerful "tool".

2 The Lamb model

The Horace Lamb model ... [7].

2.1 Physically motivated solution

...

2.2 The propagation operator

Define a Hilbert space $\mathcal{H}$ by

$$\mathcal{H} := L^2(\mathbb{R};\mathbb{C}) \oplus \mathbb{C} \tag{2.81}$$

and an operator L therein by

$$\mathcal{D}(L) = \left\{ [f^1\ f^0]^T \in H^2(\mathbb{R}^*) \oplus \mathbb{C} \,:\, f^1(0-) = f^1(0+) = f^0 \right\} \subset \mathcal{H}\,, \tag{2.82}$$

$$L \begin{bmatrix} f^1 \\ f^0 \end{bmatrix} = \begin{bmatrix} -c^2 D^2 f^1 \\ \omega_0^2\, f^0 - \beta [Df^1]_0 \end{bmatrix}, \tag{2.83}$$

in which $(Df)(x) = f'(x)$ is the derivative, $c > 0$, $\omega_0 \in \mathbb{R}$, $\beta > 0$, and $[Df]_0$ is the jump of the derivative of f, in the sense of boundary traces,

$$[g]_a = g(a+0) - g(a-0)\,. \tag{2.84}$$

In $\mathcal{H}$, define an inner product (topologically equivalent to the standard one), depending on β and c, by

$$\left\langle \begin{bmatrix} f^1 \\ f^0 \end{bmatrix}, \begin{bmatrix} g^1 \\ g^0 \end{bmatrix} \right\rangle := \beta \int \bar{f}^1\, g^1 \; + \; c^2\, \bar{f}^0 g^0\,. \tag{2.85}$$

Fact. The operator L is self-adjoint with respect to the inner product $\langle \cdot, \cdot \rangle$. This means that $\mathcal{D}(L^*) = \mathcal{D}(L)$ and that, for all $[f^1, f^0]^T$ and $[g^1, g^0]^T$ in $\mathcal{D}(L)$,

$$\left\langle \begin{bmatrix} f^1 \\ f^0 \end{bmatrix}, L \begin{bmatrix} g^1 \\ g^0 \end{bmatrix} \right\rangle = \left\langle L \begin{bmatrix} f^1 \\ f^0 \end{bmatrix}, \begin{bmatrix} g^1 \\ g^0 \end{bmatrix} \right\rangle. \tag{2.86}$$

[Exercise: Prove this.]

Computing the resolvent $(L - \omega^2 I)^{-1}$ *of* L. For each $y \neq 0$, one must solve

$$(L - \omega^2) \begin{bmatrix} G(\cdot, y; \omega) \\ G(0, y; \omega) \end{bmatrix} = \begin{bmatrix} \delta(x-y) \\ 0 \end{bmatrix},$$

which means

$$\begin{cases} -c^2\, G_{xx}(x,y;\omega) - \omega^2 G(x,y;\omega) = \delta(x-y) \qquad (x \neq 0) \\ -\beta\, [G_x(\cdot, y;\omega)]_0 + (\omega_0^2 - \omega^2) G(0,y;\omega) = 0\,. \end{cases}$$

The solution has the form of a field that is produced by a source concentrated at y and then modified, or scattered, by the resonator at $x = 0$,

$$G(x,y;\omega) = \frac{-1}{2i\,c\,\omega}\, e^{i\frac{\omega}{c}|x-y|} + g(y;\omega)\, e^{i\frac{\omega}{c}|x|}\,, \tag{2.87}$$

in which $\mathrm{Im}\,(\omega) > 0$, so that the field decays exponentially as $|x| \to \infty$. One computes that

$$g(y;\omega) = \frac{\omega^2 - \omega_0^2}{2i\,c\,\omega\,(\omega^2 + \frac{2i\beta}{c}\omega - \omega_0^2)}\, e^{i\frac{\omega}{c}|y|}. \tag{2.88}$$

One must also solve for the field produced by a source on the oscillator itself,

$$(L - \omega^2) \begin{bmatrix} G_0(\cdot;\omega) \\ G_0(0;\omega) \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

and the solution is

$$G_0(x;\omega) \;=\; \frac{-1}{\omega^2 + \frac{2i\beta}{c}\omega - \omega_0^2}\, e^{i\frac{\omega}{c}|x|}\,. \tag{2.89}$$

Now, for an element $[f^1\, f^0]^T \in \mathcal{H}$, the field $U(x,\omega)$, defined by

$$U(x,\omega) \;=\; \int f^1(y) G(x,y;\omega)\, dy + f^0 G_0(x;\omega) \tag{2.90}$$

satisfies

$$(L-\omega^2)\begin{bmatrix} U(\cdot,\omega) \\ U(0,\omega) \end{bmatrix} \;=\; \begin{bmatrix} f^1 \\ f^0 \end{bmatrix}, \tag{2.91}$$

and thus, for $\mathrm{Im}\,\omega > 0$, one has the resolvent $R(\omega, L)$ of L at ω^2

$$R(\omega,L)\begin{bmatrix} f^1 \\ f^0 \end{bmatrix} \;=\; (L-\omega^2)^{-1}\begin{bmatrix} f^1 \\ f^0 \end{bmatrix} \;=\; \begin{bmatrix} U(\cdot,\omega) \\ U(0,\omega) \end{bmatrix}. \tag{2.92}$$

2.3 The wave-equation version

In Lamb's original model, L is the propagator of a second-order equation in time,

$$\frac{d^2}{dt^2}\begin{bmatrix} u^1 \\ u^0 \end{bmatrix} = -L\begin{bmatrix} u^1 \\ u^0 \end{bmatrix}. \tag{2.93}$$

To formulate the problem as a first-order equation, introduce the velocity $[v^1, v^0]^t = d[u^1, u^0]/dt$. Writing $u = [u^1, u^0]^t$ and $v = [v^1, v^0]^t$, (2.93) becomes

$$\frac{d}{dt}\begin{bmatrix} u \\ v \end{bmatrix} = -iA\begin{bmatrix} u \\ v \end{bmatrix}, \tag{2.94}$$

in which the operator A is given by

$$A \;:=\; i\begin{bmatrix} 0 & I \\ -L & 0 \end{bmatrix}. \tag{2.95}$$

This operator is self-adjoint in $Q(L) \oplus \mathcal{H}$, where its domain is

$$\mathcal{D}(A) = \mathcal{D}(L) \oplus Q(L) \subset Q(L) \oplus \mathcal{H}\,, \tag{2.96}$$

where $Q(L)$ is the form domain of L ... (need to elaborate).

The initial-value problem for (2.93), or equivalently (2.94), stipulates an initial condition

$$u_0 = [u_0^1, u_0^0]^t, \quad v_0 = [v_0^1, v_0^0]^t\,, \tag{2.97}$$

and with the additional stipulation that the solution vanishes for $t < 0$, one obtains, as before,

$$\begin{bmatrix} u(t) \\ v(t) \end{bmatrix} \;=\; \int \begin{bmatrix} U(\omega) \\ V(\omega) \end{bmatrix} e^{-i\omega t}\, d\omega \;=\; \frac{1}{2\pi i}\int (A-\omega I)^{-1}\begin{bmatrix} u_0 \\ v_0 \end{bmatrix} e^{-i\omega t}\, d\omega\,, \tag{2.98}$$

in which the F.L. transform $[U, V]$ of $[u, v]$ is analytic in the upper half ω-plane and the integral is a limit to the real line from above.

Computing the resolvent $(A-\omega I)^{-1}$ *of* A. One has to solve

$$(A-\omega I)\begin{bmatrix} U \\ V \end{bmatrix} = \begin{bmatrix} g \\ f \end{bmatrix}, \tag{2.99}$$

which is equivalent to the pair

$$(L-\omega^2)U = (\omega g + if)\,, \tag{2.100}$$
$$V = -i(g+\omega U)\,. \tag{2.101}$$

We already have computed the kernel for the resolvent in $(L-\omega^2)^{-1}$ (2.87–2.90), and thus can compute U.

Now, to compute the solution $[u(t), v(t)]$, one takes $[g, f] = [u_0, v_0]$. Let us specialize to the initial conditions

$$u_0 = [0,0]^t\,, \quad v_0 = [0, v_0^0]^t\,, \tag{2.102}$$

which corresponds to striking an initially motionless string/mass system with an impulsive force at $t=0$ that is applied only to the mass. The solution is

$$u(x,t) = \frac{v_0^0}{2\pi}\int \frac{-1}{\omega^2 + \frac{2i\beta}{c}\omega - \omega_0^2} \exp\left[i\tfrac{\omega}{c}\left(|x|-ct\right)\right] d\omega\,. \tag{2.103}$$

The poles of the integrand are at

$$\omega_\pm = -\frac{i\beta}{c} \pm \sqrt{\omega_0^2 - \left(\frac{\beta}{c}\right)^2}\,. \tag{2.104}$$

The residue calculus gives

$$\begin{aligned} u(x,t) &= \chi_{\{|x|-ct<0\}}(x,t)\, iv_0^0 \left[\frac{\exp\left[i\frac{\omega_+}{c}\left(|x|-ct\right)\right]}{\omega_+ - \omega_-} + \frac{\exp\left[i\frac{\omega_-}{c}\left(|x|-ct\right)\right]}{\omega_- - \omega_+} \right] \\ &= \begin{cases} 0 & \text{if } |x|-ct>0\,, \\ \frac{-v_0^0}{\sqrt{\omega_0^2 - \left(\frac{\beta}{c}\right)^2}} e^{\frac{\beta}{c^2}(|x|-ct)} \sin\left(\frac{1}{c}\sqrt{\omega_0^2-\left(\frac{\beta}{c}\right)^2}\left(|x|-ct\right)\right) & \text{if } |x|-ct<0\,. \end{cases} \end{aligned} \tag{2.105}$$

Picture ...

2.4 The Schrödinger version

Replace ω_0^2 by E_0 and c^2 by μ. Now, ...

2.5 Scattering of harmonic waves

... stuff about scattering ...

In either the wave or the Schrödinger case, the harmonic problem is

$$\begin{aligned} & u'' + k^2 u = 0\,, \\ & \alpha[u']_0 + \left(k^2 - k_0^2\right) u(0) = 0\,. \end{aligned} \tag{2.106}$$

First, let's consider how a harmonic wave in the string $e^{i(kx-ct)}$, traveling from left to right, is scattered by the spring-mass attached to the line. One seeks a solution of (2.106) of the form

$$u(x) = \begin{cases} e^{ikx} + r(k)e^{-ikx} & \text{for } x<0\,, \\ t(k)e^{ikx} & \text{for } x>0\,. \end{cases} \tag{2.107}$$

The field $r(k)e^{-ik(x+ct)}$ is the *reflected field*, and $t(k)e^{ik(x-ct)}$ is the *transmitted field*. The total field $u(x)e^{-ickt}$ is called a *scattering state* or *extended state*. It can be parsed as sum of the *incident field* $e^{ik(x-ct)}$ and the *scattered field*,

$$u_{\text{sc}}(x) = \begin{cases} r(k)e^{-ikx} & \text{for } x<0\,, \\ \left(t(k)-1\right)e^{ikx} & \text{for } x>0\,. \end{cases} \tag{2.108}$$

Because of the symmetry of the Lamb model, the scattering coefficients r and t for an incident wave in the string $e^{-i(kx+ct)}$ traveling from right to left are identical to those for a right-traveling incident field. Thus one has the scattering state

$$u(x) = \begin{cases} t(k)e^{-ikx} & \text{for } x < 0\,, \\ e^{-ikx} + r(k)e^{ikx} & \text{for } x > 0\,. \end{cases} \tag{2.109}$$

By linearly combining the scattering states (2.107) and (2.109), one obtains states of the form

$$u(x) = \begin{cases} j_+e^{ikx} + u_-e^{-ikx} & \text{for } x < 0\,, \\ u_+e^{ikx} + j_-e^{-ikx} & \text{for } x > 0\,. \end{cases} \tag{2.110}$$

Because of the linearity of the Lamb system, one obtains

$$\begin{bmatrix} u_+ \\ u_- \end{bmatrix} = \begin{bmatrix} t(k) & r(k) \\ r(k) & t(k) \end{bmatrix} \begin{bmatrix} j_+ \\ j_- \end{bmatrix} = S(k) \begin{bmatrix} j_+ \\ j_- \end{bmatrix}. \tag{2.111}$$

The matrix $S(k)$ is called the *scattering matrix* or the S-matrix. The coefficients $j_\pm$ are the complex amplitudes of the *incoming* field, and the coefficients $j_\pm$ are the complex amplitudes of the *outgoing field.* The incident field is considered to be the field $u_{\text{inc}}(x) = j_+e^{ikx} + j_-e^{-ikx}$, and the scattered field is, as above, defined through

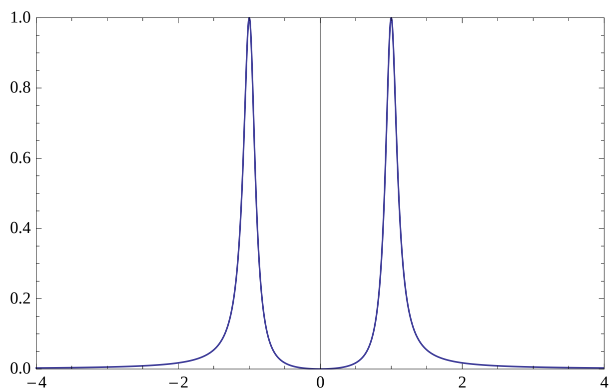
$$u = u_{\text{inc}} + u_{\text{sc}} \qquad \text{for } x \in \mathbb{R}. \tag{2.112}$$

Notice that the scattered field satisfies an *outgoing condition*, namely, that it is a multiple of e^{ikx} (a rightward wave) for $x > 0$ and a multiple of e^{-ikx} (a leftward wave) for $x < 0$. For real k, this condition is also called the *radiation condition.*

S is unitary ... prove it by integration by parts.

The radiation/outgoing condition ...

The transmission coefficient and the limit as $\beta \to 0$—transmission resonances.



3 An elaborated Lamb model

... anybody volunteer to write this up?

4 Resonance in a discrete model

We now consider an extended (ambient) medium that is discrete, and still one-dimensional, namely the integer lattice. ...

4.1 The second-difference operator in the 1D lattice

The group of integers $\mathbb{Z}$ acts as a translational symmetry group on the integer lattice. This induces a unitary group of operators on $\ell^2(\mathbb{Z})$ isomorphic to $\mathbb{Z}$:

$$g : u \mapsto u(\cdot + g). \tag{4.113}$$

This translation group is generated by the shift operator

$$S : u \mapsto u(\cdot + 1) \tag{4.114}$$

through composition and inversion of maps. For each unitary number e^{ik} ($k \in \mathbb{R}$), this action has a (non-ℓ^2) eigenfunction, or *character*

$$\chi_k(n) = e^{ikn} \tag{4.115}$$

with eigenvalue e^{ik}, that is,

$$S\chi_k = e^{ik}\chi_k\,. \tag{4.116}$$

These characters provide a concrete realization of the spectral resolution of ℓ^2 for the shift operator, which we know as the Fourier transform $\mathcal{F}$. To wit: each $f \in \ell^2(\mathbb{Z})$ can be written as an integral superposition of characters; the complex amplitudes are given by

$$\hat{f}(k) = (\mathcal{F}f)(k) := \sum_{g\in\mathbb{Z}} f(g)e^{-ikg}\,, \tag{4.117}$$

and the function f is recovered by

$$f(n) = \left(\mathcal{F}^{-1}\hat{f}\right)(n) := \frac{1}{2\pi}\int_{-\pi}^{\pi} \hat{f}(k)e^{ikn}dk\,. \tag{4.118}$$

The Fourier transform is a unitary operator from $\ell^2(\mathbb{Z})$ to $L^2[-\pi,\pi]$, and it diagonalizes the shift operator; that is, it converts S into a multiplication operator on $L^2[-\pi,\pi]$:

$$(Sf)\hat{}(k) = e^{ik}\hat{f}(k)\,. \tag{4.119}$$

The centered second-difference operator D^2 on $\ell^2(\mathbb{Z})$,

$$(D^2f)(n) = f(n+1) - 2f(n) + f(n-1)\,, \tag{4.120}$$

commutes with the shift operator S; in fact, it is a function of S,

$$D^2 = S - 2I + S^{-1}\,. \tag{4.121}$$

Thus D^2 is also diagonalized by the Fourier transform,

$$(D^2f)\hat{}(k) = \left(e^{ik} - 2 + e^{-ik}\right)\hat{f}(k) = 2(\cos k - 1)\hat{f}(k)\,. \tag{4.122}$$

Since $2(\cos k - 1)$ is negative for $k \in \mathbb{R}$, D^2 is a negative operator. Set

$$L := -D^2 = 2I - \left(S + S^{-1}\right)\,, \tag{4.123}$$

which is represented on $L^2[-\pi,\pi]$ by the operator of multiplication by the function

$$\hat{m}(k) = 2\,(1 - \cos k) = 4\sin^2\tfrac{k}{2}\,. \tag{4.124}$$

Since $\hat{m}$ is bounded, so is L, and since $\hat{m}$ is positive, so is L. The resolvent $(L - \lambda I)^{-1}$ of L is represented by multiplication by $(\hat{m} - \lambda)^{-1}$ whenever it is a bounded operator from $L^2[-\pi,\pi]$ to itself. This occurs exactly for those λ not in the range $[0,4]$ of $\hat{m}$. Thus the spectrum of L is

$$\sigma(L) = [0,4]\,. \tag{4.125}$$

Since χ_k is a (non-ℓ^2) eigenfunction of S, with eigenvalue e^{ik}, it is also an eigenfunction of $L = 2I - S - S^{-1}$ with eigenvalue $2(1-\cos k)$. Thus the spectrum of L consists of all λ for which there exists $k \in \mathbb{R}$ such that χ_k satisfies

$$L\chi_k = \lambda\chi_k\,. \tag{4.126}$$

The characters χ_k ($k \in \mathbb{R}$) are called the *extended states* for L (or radiating states, generalized eigenfunctions, propagating modes, oscillatory modes, etc.). For each $\lambda \in (0,4)$, there are two such states

$$\chi_{\pm k}(n) = e^{\pm ikn}\,. \tag{4.127}$$

The locus of the relation $\lambda = 2(1-\cos k)$ is called the "Fermi surface" for L. If one introduces time dynamics through $i\partial_t u(n,t) = Lu(n,t)$ or $\partial_{tt}u = -Lu$, one has harmonic solutions of the form

$$\chi_k(n)e^{-i\omega t} = e^{i(kn-\omega t)} \tag{4.128}$$

for each pair (k,ω) for which $\omega = 2(1-\cos k)$ (in the case $i\partial_t$) or $\omega^2 = 2(1-\cos k)$ (in the case ∂_{tt}). This relation between frequency ω and wavenumber k is called the *dispersion relation* for the dynamical system. It has period 2π in k.

The resolvent of L. To compute the kernel for the resolvent of L for $\lambda \notin [0,4]$, let k be the solution of $\lambda = 2(1-\cos k)$ such that $\operatorname{Im} k > 0$. Then notice that the function $e^{ik|\cdot|}$ is exponentially decaying and satisfies

$$\left((L-\lambda I)e^{ik|\cdot|}\right)(n) = \delta(n)\left[2(1-e^{ik})-\lambda\right]\,, \tag{4.129}$$

in which δ is the function such that $\delta(0) = 1$ and $\delta(n) = 0$ for all $n \neq 0$. Thus, the function $u = e^{ik|\cdot|}/\left[2(1-e^{ik})-\lambda\right]$ satisfies $(L-\lambda I)u = \delta$. By shifting the forcing to the lattice point m one obtains the solution $G(n,m;k)$ of $(L-\lambda I)u = \delta(\cdot - m)$,

$$G(n,m;k) = \frac{e^{ik|n-m|}}{2(1-e^{ik})-\lambda} \qquad \big(\lambda = 2(1-\cos k)\big)\,. \tag{4.130}$$

So $G(n,m;k)$, as a function of n, is the λ-harmonic response to a unit forcing at the lattice point m. One then obtains the solution to the forced problem

$$(L-\lambda I)u = f \tag{4.131}$$

for any forcing function $f \in \ell^2(\mathbb{Z})$ by writing $f(n) = \sum_{m\in\mathbb{Z}} f(m)\delta(n-m)$ and using linear superposition:

$$u(n) = (L-\lambda I)^{-1}f(n) = \sum_{m\in\mathbb{Z}} G(n,m;k)f(m)\,. \tag{4.132}$$

4.2 Analytic continuation of the resolvent into the lower half k-plane.

For $\operatorname{Im} k > 0$, the resolvent of L is a bounded operator in ℓ^2; let us denote it by $R_0(k)$:

$$R_0(k) : \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z}) \qquad (\operatorname{Im} k > 0)\,, \tag{4.133}$$

$$R_0(k)f(n) = \sum_{m\in\mathbb{Z}} \frac{e^{ik|m-n|}}{2(1-e^{ik})-\lambda} f(m)\,. \tag{4.134}$$

Evidently, this action can be extended meromorphically to $k \in \mathbb{C}$ as an operator on a restricted domain but having a larger range:

$$R_0(k) : \ell^2_0(\mathbb{Z}) \to \mathfrak{F}(\mathbb{Z}) \qquad (k \in \mathbb{C} \setminus \{2\pi\mathbb{Z}\})\,, \tag{4.135}$$

in which $\ell^2_0(\mathbb{Z})$ consists of all functions in $\ell^2(\mathbb{Z})$ that have bounded support and $\mathfrak{F}(\mathbb{Z}) = \mathbb{C}^{\mathbb{Z}}$ is the vector space of all complex-valued functions on $\mathbb{Z}$.

R_0 is *unbounded* in the ℓ^2 norm for $\operatorname{Im} k \leq 0$. The sense in which R_0 is analytic is that of pointwise evaluation, that is, $(R_0(k)f)(n)$ is analytic as a function of k off the pole set $2\pi\mathbb{Z}$. $R_0(k)$ is periodic with period 2π.

4.3 The discrete wave equation

Consider the dynamical system for an evolving function $u(n,t)$ in $\ell^2(\mathbb{Z})$, starting from rest and impulsively attaining an initial velocity of $v_0(n)$ at $t=0$,

$$u_{tt} = -Lu + v_0(n)\delta(t)\,, \tag{4.136}$$

$$u(n,t) = 0 \quad \text{for } t<0\,. \tag{4.137}$$

The Fourier-Laplace transform $U(n,\omega)$ of $u(n,t)$ satisfies

$$U = \frac{1}{2\pi}(L-\omega^2)^{-1}v_0\,, \tag{4.138}$$

or, using the kernel for the resolvent derived above,

$$U(n,\omega) = \frac{1}{2\pi}\sum_{m\in\mathbb{Z}} \frac{e^{ik|n-m|}}{2(1-e^{ik})-\omega^2}\, v_0(m)\,. \tag{4.139}$$

Let us concentrate the impulsive force at the 0^{th} lattice site,

$$v_0(n) = \delta_{0n}\,, \tag{4.140}$$

so that

$$U(n,\omega) = \frac{1}{2\pi}\frac{e^{ik|n|}}{2(1-e^{ik})-\omega^2}\,, \tag{4.141}$$

and

$$u(n,t) = \frac{1}{2\pi}\int_{-\infty+0i}^{\infty+0i} \frac{e^{i(k|n|-\omega t)}}{2(1-e^{ik})-\omega^2}d\omega \qquad (\operatorname{Im}k>0 \text{ for } \operatorname{Im}\omega>0)\,. \tag{4.142}$$

By contour integration and carefully checking the behavior of the integrand in its complex domain $\mathbb{C}\setminus[-2,2]$, one obtains that $u(n,t)=0$ for $t<0$, in accordance with the assumption, and that, for $t>0$,

$$u(n,t) = \frac{1}{2\pi}\int_{-2+0i}^{2+0i} + \int_{2-0i}^{-2-0i} \frac{e^{i(k|n|-\omega t)}}{2(1-e^{ik})-\omega^2}d\omega \tag{4.143}$$

$$= \frac{1}{2\pi}\int_{-2+0i}^{2+0i} \frac{e^{i(k|n|-\omega t)}}{2(1-e^{ik})-\omega^2}d\omega + \frac{1}{2\pi}\int_{2+0i}^{-2+0i} \frac{e^{i(-k|n|-\omega t)}}{2(1-e^{-ik})-\omega^2}d\omega \tag{4.144}$$

$$= \frac{1}{2\pi}\int_{-2+0i}^{2+0i} \frac{e^{i(k|n|-\omega t)}}{2(1-e^{ik})-\omega^2}d\omega - \frac{1}{2\pi}\int_{-2+0i}^{2+0i} \frac{e^{i(k|n|+\omega t)}}{2(1-e^{ik})-\omega^2}d\omega \tag{4.145}$$

$$= \frac{i}{4\pi}\int_{-\pi}^{\pi} \frac{e^{i(k|n|-\omega t)}\cos\frac{k}{2}}{\sin k}dk - \frac{i}{4\pi}\int_{-\pi}^{\pi} \frac{e^{i(k|n|+\omega t)}\cos\frac{k}{2}}{\sin k}dk\,. \tag{4.146}$$

Now switch the branch cut to a pair of cuts along $(-\infty,-2]$ and $[2,\infty)$ so that the integrand is analytic across the spectrum $[-2,2]$ in ω and so that the range of k values in the relation $\omega = 2\sin\frac{k}{2}$ is the strip $(-\pi,\pi)\times\mathbb{R}$. [Need picture.] By contour integration [Need details], one finds that the second integral in (4.146) vanishes, so that

$$u(n,t) = \frac{i}{4\pi}\int_{-\pi}^{\pi} \frac{e^{i(k|n|-\omega t)}\cos\frac{k}{2}}{\sin k}dk, \tag{4.147}$$

which expresses the solution as an integral superposition of outward-going waves. This integral can be written as a superposition of normal modes by contour integration and residue calculus (the only pole of the denominator is at $k=0$),

$$u(n,t) = \frac{1}{2} + \frac{i}{4\pi}\int_C \frac{e^{i(k|n|-\omega t)}\cos\frac{k}{2}}{\sin k}dk\,, \tag{4.148}$$

in which C is a contour starting at $-\pi$, running vertically downward to $-\pi-ib$, then horizontally to $\pi-ib$, and then vertically upward to π [Need picture].

5 Resonant opening of gaps in spectral bands by decoration

Example of graph operators from [11] ...

Specialize to $L = -D^2$ from above and a simple decoration.

[1, 2, 3, 4, 5, 6, 7, 9, 10, 8, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21]

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