Math 7311, Fall 2015
Problem Set 1

1. Let $\{x_n\}_{n=1}^{\infty}$ be a sequence from $\mathbb{R}$. Prove that

$$\liminf_{n\to\infty} x_n \leq \limsup_{n\to\infty} x_n$$

and that equality holds if and only if $\lim_{n\to\infty} x_n$ exists, in which case all three limits are equal.

2. Let $\{x_n\}_{n=1}^{\infty}$ be a sequence from $\mathbb{R}$. Prove that

$$\limsup_{n\to\infty} x_n = \sup \limsup_{n\to\infty} (-\infty, x_n].$$

Here, $\limsup_{n\to\infty} (-\infty, x_n]$ is a set, as we have defined it, and the supremum of a set is its least upper bound.
[BTW: Related statements hold: $\liminf_{n\to\infty} x_n = \sup \liminf_{n\to\infty} (-\infty, x_n]$, and things change around if one uses the intervals $[x_n, \infty)$. The intervals may also be open instead of closed.]

3. Let a function $f : X \to Y$ be given. Prove that, for each indexed family $\{X_\alpha\}_{\alpha\in A}$ of subsets of X,

$$f\Big(\bigcap_{\alpha\in A} X_\alpha\Big) \subset \bigcap_{\alpha\in A} f(X_\alpha),$$

and that equality holds for each $\{X_\alpha\}_{\alpha\in A}$ if and only if f is injective.

4. Prove that the axiom of choice is equivalent to the following statement: Every surjective function admits a right inverse; that is, if $f : X \to Y$ is surjective, then there exists a function $g : Y \to X$ such that $f \circ g\,(y) = y$ for each $y \in Y$.

5. Define the following linear ordering on $\mathcal{P}(\mathbb{N})$: If X_1 and X_2 are distinct subsets of $\mathbb{N}$, set

$$m = \min\{n \in \mathbb{N} : n \in X_1 \Delta X_2\},$$

and then set $X_i \leq X_j$ if $m \in X_i$ (this is a kind of lexicographic ordering). Prove that $\mathcal{P}(\mathbb{N})$ is not well ordered under this ordering.