Ch. 1, #5.

Set $\mathcal{M} = \mathcal{M}(\mathcal{E})$, the σ-algebra generated by $\mathcal{E}$, and set

$$\mathcal{N} = \cup \{\mathcal{M}(\mathcal{F}) : \mathcal{F} \subset \mathcal{E},\ |\mathcal{F}| \leq |\mathbb{N}|\}.$$

For $\mathcal{F} \subset \mathcal{E}$, $\mathcal{M}(\mathcal{F}) \subset \mathcal{M}(\mathcal{E}) = \mathcal{M}$, and thus $\mathcal{N} \subset \mathcal{M}$. $\mathcal{N}$ is closed under complements because for all $A \in \mathcal{N}$, there is a countable subset $\mathcal{F}$ of $\mathcal{E}$ such that $A \in \mathcal{M}(\mathcal{F}) \subset \mathcal{N}$ and thus $A^c \in \mathcal{M}(\mathcal{F}) \subset \mathcal{N}$ also. To show that $\mathcal{N}$ is closed under countable unions, let $\mathcal{H}$ be a countable subset of $\mathcal{N}$. For each $H \in \mathcal{H}$, there is a countable subset $\mathcal{F}_H$ of $\mathcal{E}$ such that $H \in \mathcal{M}(\mathcal{F}_H)$. Since $\mathcal{M}(\mathcal{F}_H) \subset \mathcal{M}\left(\cup_{H\in\mathcal{H}}\mathcal{F}_H\right)$, we have $\mathcal{H} \subset \mathcal{M}\left(\cup_{H\in\mathcal{H}}\mathcal{F}_H\right)$. Since $\cup_{H\in\mathcal{H}}\mathcal{F}_H$ is countable, $\mathcal{M}(\cup_{H\in\mathcal{H}}\mathcal{F}_H)$ is included in $\mathcal{N}$ and one therefore has $\cup_{H\in\mathcal{H}}H \subset \mathcal{M}(\cup_{H\in\mathcal{H}}\mathcal{F}_H) \subset \mathcal{N}$.

Ch. 1, #8.

For the first inequality,

$$\begin{aligned}\mu\left(\liminf E_j\right) &= \mu\left(\cup_{j\geq 1}\cap_{i\geq j} E_i\right) = \lim_{j\to\infty}\mu\left(\cap_{i\geq j}E_i\right)\\ &\leq \lim_{j\to\infty}\inf_{i\geq j}\mu(E_i) = \liminf_{j\to\infty}\mu(E_j).\end{aligned} \tag{0.1}$$

The first equality is by definition and the second by lower continuity of measures. The inequality holds because $\cap_{i\geq j}E_i \subset \mu(E_j)$ for all j and the monotonicity of measures. For the second inequality,

$$\begin{aligned}\mu\left(\limsup E_j\right) &= \mu\left(\cap_{j\geq 1}\cup_{i\geq j} E_i\right) = \lim_{j\to\infty}\mu\left(\cup_{i\geq j}E_i\right)\\ &\geq \lim_{j\to\infty}\sup_{i\geq j}\mu(E_i) = \limsup_{j\to\infty}\mu(E_j).\end{aligned} \tag{0.2}$$

The justifications are analogous to the previous case, except that one needs to invoke the finiteness of $\cup_{i\geq j}E_i$ for some j for the second equality.

Ch. 1, #12.

(a) For $E, F \in \mathcal{M}$, suppose that $\mu(E\triangle F) = 0$, so that

$$\mu(E) = \mu(E\cap F) + \mu(E\cap F^c) \leq \mu(E\cap F) + \mu(E\triangle F) = \mu(E\cap F). \tag{0.3}$$

The first equality is due to disjoint additivity of measures, and the second comes from $E\cap F^c \subset E\triangle F$ and the monotonicity of measures. By a symmetric argument, one has $\mu(F) = \mu(E\cap F)$, so that $\mu(E) = \mu(F)$.

(b) Reflexivity: $\mu(E\triangle E) = \mu(\emptyset) = 0$, so $E \sim E$. Symmetry stems from the symmetry of the symmetric difference: $E\triangle F = F\triangle E$. Transitivity: Suppose $E \sim F$ and $F \sim G$, so $\mu(E\triangle F) = \mu(F\triangle G) = 0$. We have $E\triangle G = (E\cap G^c)\mathring{\cup}(G\cap E^c)$ and

$$E\cap G^c = (E\cap(F\cap G^c))\mathring{\cup}((E\cap F^c)\cap G^c), \tag{0.4}$$

and thus

$$\mu(E\cap G^c) \leq \mu(F\cap G^c) + \mu(E\cap F^c) \leq \mu(F\triangle G) + \mu(E\triangle F) = 0. \tag{0.5}$$

By a symmetric argument, one obtains $\mu(G\cap E^c) = 0$, and thus $\mu(E\triangle G) = 0$.

(c) First, we show that ρ is well defined on equivalence classes of measurable sets. Suppose $E_1 \sim E_2$ and $F_1 \sim F_2$. We must show that $\mu(E_1 \triangle F_1) = \mu(E_2 \triangle F_2)$. One has

$$\begin{aligned} E_1 \triangle F_1 &= (E_1 \cap F_1^c) \mathring{\cup} (F_1 \cap E_1^c) \\ &= (E_1 \cap F_1^c \cap F_2) \mathring{\cup} (E_1 \cap F_1^c \cap F_2^c) \mathring{\cup} (F_2 \cap F_1 \cap E_1^c) \mathring{\cup} (F_2^c \cap F_1 \cap E_1^c) \end{aligned}$$

and therefore

$$\begin{aligned} \mu(E_1 \triangle F_1) &\leq \mu(E_1 \cap F_1^c \cap F_2^c) + \mu(E_1^c \cap F_1 \cap F_2) + \mu(E_1 \cup (F_1 \triangle F_2)) + \mu(E_1^c \cap (F_2^c \triangle F_1)) \\ &= \mu(E_1 \cap F_1^c \cap F_2^c) + \mu(E_1^c \cap F_1 \cap F_2). \end{aligned}$$

By replacing F_1 by F_2, the latter expression is also equal to $\mu(E_1 \triangle F_2)$ so that $\mu(E_1 \triangle F_1) = \mu(E_1 \triangle F_2)$. By an analogous argument, one obtains $\mu(E_1 \triangle F_2) = \mu(E_2 \triangle F_2)$.

Now we show that $\rho(E, F) = \mu(E \triangle F)$ is a metric on $\mathcal{M}$. $\rho(E, E) = \mu(E \triangle E) = \mu(\emptyset) = 0$; and if $\rho(E, F) = 0$, then $\mu(E \triangle F) = 0$, so $E \sim F$, that is, the equivalence classes of E and F are the same. Symmetry is a result of the symmetry of the symmetric difference. To show the triangle inequality, observe that

$$E \cap G^c = (E \cap G^c \cap F) \;\cup\; (E \cap G^c \cap F^c) \subset (G^c \cap F) \;\cup\; (E \cap F^c), \tag{0.6}$$

and, similarly, that $E^c \cap G \subset (E^c \cap F) \;\cup\; (G \cap F^c)$. thus $E \triangle G \subset G \triangle F \cup E \triangle F$, which implies $\rho(E, G) = \mu(E \triangle G) \leq \mu(E \triangle F) + \mu(F \triangle G) = \rho(E, F) + \rho(F, G)$.

Ch. 1, #18.

(a) By definition,

$$\mu^*(E) \;=\; \inf\Big\{\sum_{i=0}^{\infty} \mu_0(A_i) : A_i \in \mathcal{A},\, E \subset \cup_{i=1}^{\infty} A_i\Big\}, \tag{0.7}$$

and thus, for any $\epsilon > 0$, there is a collection of sets $A_i \in \mathcal{A}$ covering E such that

$$\sum_{i=0}^{\infty} \mu_0(A_i) \leq \mu^*(E) + \epsilon. \tag{0.8}$$

Set $A = \cup_{i=1}^{\infty} A_i \in \mathcal{A}_\sigma$ so that $E \in A$ and

$$\mu^*(A) \leq \sum_{i=0}^{\infty} \mu_0(A_i) \leq \mu^*(E) + \epsilon. \tag{0.9}$$

(b) Suppose that E is μ^*-measurable with $\mu^*(E) < \infty$. From part (a), there are sets $A_i \in \mathcal{A}_\sigma$ such that $E \subset A_i$ and $\mu^*(A_i) \leq \mu^*(E) + i^{-1}$. Set $A := \cap_{i=1}^{\infty} A_i \in \mathcal{A}_{\sigma\delta}$. Since the A_i are measurable (Theorem 1.14) and $\mu^*(A_1) \leq \mu^*(E) + 1 < \infty$, upper continuity holds:

$$\mu^*(A) = \mu^*(\cap_{i=1}^{\infty} A_i) = \lim_{n\to\infty} \mu^*(\cap_{i=1}^{n} A_i) \leq \lim_{n\to\infty} \mu^*(A_n) = \mu^*(E). \tag{0.10}$$

Since $E \subset A$, this inequality yields $\mu^*(E) = \mu^*(A)$. By disjoint additivity of measures (μ^* restricted to the μ^*-measurable sets) $\mu^*(A \setminus E) = \mu^*(A) - \mu^*(E) = 0$.

Suppose now that $\mu^*(A \setminus E) = 0$ with $A \in \mathcal{A}_{\sigma\delta}$. First we show that $A \setminus E$ is measurable. In fact, for any $B \subset X$, if $\mu^*(B) = 0$, then B is measurable. To wit: For each $F \subset X$,

$$\mu^*(F \cap B) + \mu^*(F \cap B^c) \leq \mu^*(B) + \mu^*(F) = \mu^*(F). \tag{0.11}$$

The inequality comes from the monotonicity of outer measures, and the equality comes from $\mu^*(B) = 0$. Since $A \in \mathcal{A}_{\sigma\delta} \subset \mathcal{M}$, A is measurable and therefore $E = A \setminus (A \setminus E)$ is measurable.

(c) Since μ_0 is sigma-finite, one can write X as a disjoint union of sets $X_k \in \mathcal{A}$ with $\mu_0(X_k) < \infty$,

$$X = \mathring{\bigcup}_{k\in\mathbb{N}} X_k. \tag{0.12}$$

Since $E \subset X$ is measurable, $E_k := E \cap X_k$ is also measurable, and one has

$$E = \mathring{\bigcup}_{k\in\mathbb{N}} E_k. \tag{0.13}$$

According to part (b), there are sets $A_k \in A_{\sigma\delta}$ such that $E_k \subset A_k$ and $\mu^*(A_k \setminus E_k) = 0$. Let

$$A_k = \bigcap_{j\in\mathbb{N}} \bigcup_{i\in\mathbb{N}} A_{ijk}, \tag{0.14}$$

in which the sets A_{ijk} are in $\mathcal{A}$. Since $E_k \subset X_k$, one may assume that $A_{ijk} \subset X_k$ by replacing A_{ijk} by $A_{ijk} \cap X_k \in \mathcal{A}$. Set

$$A := \mathring{\bigcup}_{k\in\mathbb{N}} A_k. \tag{0.15}$$

One has $E \subset A$ with $\mu^*(A \setminus E) \leq \sum_{k\in\mathbb{N}} \mu^*(A_k \setminus E_k) = 0$. Since $A_{i_1j_1k_1} \cup A_{i_2j_2k_2} = \emptyset$ for $k_1 \neq k_2$, one has

$$A = \mathring{\cup}_{k\in\mathbb{N}} A_k = \cup_{k\in\mathbb{N}} \cap_{j\in\mathbb{N}} \cup_{i\in\mathbb{N}} A_{ijk} = \cap_{j\in\mathbb{N}} \cup_{i,k\in\mathbb{N}} A_{ijk}. \tag{0.16}$$

Since $\cup_{i,k\in\mathbb{N}} A_{ijk} \in \mathcal{A}$ for each $j \in \mathbb{N}$, one has $A \in \mathcal{A}_{\sigma\delta}$.

Ch. 1, #28.

Ch. 1, #30.

Ch. 1, #31.

Given $\alpha : 1/2 < \alpha < 1$, there is, by problem ..., an open interval I such that $m(E \cap I) \geq \alpha\, m(I)$. Let $I = (x_1, x_2)$. Let $d : 0 < d < m(I)$ be given, and set $I_1 = (x_1, x_2 - d)$. Suppose that $d \notin E - E$. Then for each $x \in I_1 \cap E$, one has $x + d \notin E$, that is, $((I_1 \cap E) + d) \cap ((I_1 + d) \cap E) = \emptyset$. Thus we have a disjoint union

$$((I_1 \cap E) + d) \mathring{\cup} ((I_1 + d) \cap E) \;\subset\; I_1 + d, \tag{0.17}$$

whence

$$\mu((I_1 \cap E) + d) + \mu((I_1 + d) \cap E) \leq \mu(I_1 + d). \tag{0.18}$$

Since $(E \cap I) \subset (I_1 \cap E) \mathring{\cup} (x_2 - d, x_2)$, one has $\mu(I \cap E) \leq \mu(I_1 \cap E) + d$ and thus

$$\mu(I_1 \cap E) \;\geq\; \mu(E \cap I) - d \;\geq\; \alpha\, m(I) - d, \tag{0.19}$$

and, likewise,

$$\mu((I_1 + d) \cap E) \;\geq\; \mu(E \cap I) - d \;\geq\; \alpha m(I) - d. \tag{0.20}$$

Therefore, by (0.18-0.20) and translation invariance of Lebesgue measure

$$\begin{aligned} \mu(I) - d = \mu(I_1 + d) &\geq \mu((I_1 \cap E) + d) + \mu((I_1 + d) \cap E) \\ &= \mu(I_1 \cap E) + \mu((I_1 + d) \cap E) \\ &\geq 2(\alpha\, \mu(I) - d), \end{aligned} \tag{0.21}$$

and this implies

$$(2\alpha - 1)\mu(I) \;\leq\; d \qquad \text{if } \; d \notin E - E. \tag{0.22}$$

By contraposition, if $d < (2\alpha - 1)\mu(I)$, then $d \in E - E$, that is there are x and y in E such that $x - y = d$; therefore also, $-d = y - x \in E - E$. It follows that $E - E$ contains the open interval

$$\big(-(2\alpha - 1)\mu(I),\; (2\alpha - 1)\mu(I) \big). \tag{0.23}$$

Ch. 1, #33.