(1)

Spectral Theory in Wave Dynamics

Math 7384
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eqns, structure
· vibrations
⇒ e-vals
· why spec
· why harmo
· why cx an.

Some linear wave equations:

acoustics: small-amp disturbances in uniform medium:
p: pressure, $\vec{u}$: velocity $= (u_1, u_2, u_3)$

$$\begin{cases} \dfrac{\partial p}{\partial t} + k\nabla\cdot u = 0 \\ \dfrac{\partial u}{\partial t} + \dfrac{1}{\rho}\nabla p = 0 \end{cases}$$

$$\Rightarrow \quad \frac{\partial^2 p}{\partial t^2} - \frac{k}{\rho}\nabla\cdot\nabla p = 0$$

$$\frac{\partial^2 u}{\partial t^2} - \frac{k}{\rho}\nabla\nabla\cdot u = 0$$

Id: $\nabla\times(\nabla\times u) = \nabla(\nabla\cdot u) - \nabla\cdot\nabla u$

so $\nabla\times u = 0 \Rightarrow \dfrac{\partial^2 u}{\partial t^2} - \dfrac{k}{\rho}\nabla\cdot\nabla u = 0$

$\Rightarrow$ Wave equation for scalar: $\dfrac{\partial^2\phi}{\partial t^2} - c^2\nabla^2\phi = 0 \quad (\nabla^2 = \Delta$: Laplacian

Harmonic version: $\phi \mapsto \phi(x)e^{i\omega t} \Rightarrow \nabla^2\phi + \left(\frac{\omega}{c}\right)^2\phi = 0$

· equiv. to $\dfrac{\partial}{\partial t}\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 0 & I \\ c^2\Delta & 0 \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} \sim$

Show $A = \begin{bmatrix} 0 & I \\ c^2\Delta & 0 \end{bmatrix}$ is (formally) anti-self-adj. w.r.to correct $\langle\, ,\rangle$.

· equiv. to $\dfrac{\partial}{\partial t}\begin{bmatrix} p \\ u \end{bmatrix} = -\begin{bmatrix} 0 & k\nabla\cdot \\ \frac{1}{\rho}\nabla & 0 \end{bmatrix}\begin{bmatrix} p \\ u \end{bmatrix}$

Show operator is anti-self-adj. w/r to $\langle\, ,\rangle_{L^2_{k^{-1}}(\mathbb{R})} + \langle\, ,\rangle_{L^2_\rho(\mathbb{R}^3)}$

electromagnetics : $\frac{1}{c}\frac{\partial}{\partial t}\begin{bmatrix}E\\ H\end{bmatrix} = \begin{bmatrix}0 & \frac{1}{\varepsilon}\nabla\times \\ -\frac{1}{\mu}\nabla\times & 0\end{bmatrix}\begin{bmatrix}E\\ H\end{bmatrix}$ w/ $\begin{cases}\nabla\cdot\varepsilon E = 0\\ \nabla\cdot\mu H = 0\end{cases}$

Solve for H : $\frac{1}{c^2}\frac{\partial^2}{\partial t^2}H = -\frac{1}{\mu}\nabla\times\frac{1}{\varepsilon}\nabla\times H$

for E : $\frac{1}{c^2}\frac{\partial^2}{\partial t^2}E = -\frac{1}{\varepsilon}\nabla\times\frac{1}{\mu}\nabla\times E$

See Kuchment "Math of PCs" for more on the operator of Maxwell.

elasticity $\rho\frac{\partial^2 u}{\partial t^2} = \nabla\cdot\sigma$ (+ forces)

$\sigma = K\,\tfrac{1}{2}(\nabla u + (\nabla u)^T)$,

$\sigma = K\varepsilon$

quantum mechanics ~~$i\psi_t$~~ $i\frac{\partial}{\partial t}\psi = -\Delta\psi + q(x)\psi$

Common structure : $\frac{d}{dt}u = Au$, A anti-self-adjoint for cons., osc. w/r. to $(\cdot,\cdot)$

$\Rightarrow$ conservation of $\|\cdot\|$:

$$\frac{d}{dt}(u,u) = (\dot u,u)+(u,\dot u) = (Au,u)+(u,Au) = 0$$

Math involved $\longrightarrow$ spectral theory, harmonic analysis, complex analysis, (moment problems)

- Why spectral theory?

If A is transl.-invariant (in time), $\frac{d}{dt}$ & A commute (A acting in H)

So seek simultaneous efcns of $\frac{d}{dt}$ & A.

$$\frac{d}{dt}e^{\lambda t} = \lambda e^{\lambda t} \qquad Au_\lambda = \lambda u_\lambda \quad \rightsquigarrow \quad \frac{d}{dt}(u_\lambda e^{\lambda t}) = A(u_\lambda e^{\lambda t})$$

Superposition principle : $u(t) = e^{At}u(0) = Me^{\Lambda t}M^{-1}u(0)$.

- Why harmonic analysis?

One needs a <u>concrete</u> spectral theory to decompose A:

Use symmetries of A (& other stuff)

<u>ex</u> A const-coeff. diff op. on $\mathbb{R}^n$ or other convolution op.

$\Rightarrow$ A commutes w/ $\mathbb{R}^n$ action of translations $T_x(u) = u(\cdot - x)$

$$T_x A = A T_x \quad \forall\ x \in \mathbb{R}^n$$

$\Leftrightarrow$ T_x are unitary op. in L^2 w/ eigenfcns $e^{2\pi i \xi \cdot x} = \chi_\xi(x)$

$$T_y \chi_\xi(x) = \chi_\xi(x+y) = \underline{2\pi i \xi \cdot y}\ \chi_\xi(x)$$

$$f \in L^2(\mathbb{R}^n) \Rightarrow f(x) = \int_{\mathbb{R}^n} \hat{f}(\xi) e^{2\pi i \xi x} d\xi$$

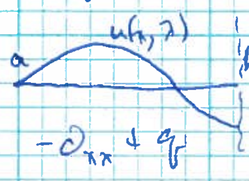
$$\& \ (T_y f)(x) = \int_{\mathbb{R}^n} 2\pi i \xi \cdot y\ \hat{f}(\xi)\, e^{2\pi i \xi x} d\xi$$

$$(Af)(x) = \int_{\mathbb{R}^n} \hat{A}(\xi) \hat{f}(\xi)\, e^{2\pi \xi x} d\xi \quad \text{— diagonalization}$$

- Why complex analysis
  - Computing $\int \hat{A}(\xi) \hat{f}(\xi)\, e^{2\pi i \xi x} d\xi$ by residue calculus ($e^{\hat{A}(\xi) t}$)

    $\Rightarrow$ normal-mode representation of solutions

  - Weyl M-function $M(\lambda)$ & friends/relatives



$$M(\lambda) = \frac{u(b,\lambda)}{u'(b,\lambda) + \alpha u'(b,\lambda)} \quad : \text{takes UHP to itself}$$

Examples of e.val problems in math physics

* Schrodinger eq. on the line w/ local well potential.
* evals of a sphere ~ spherical harmonics
* Helmholtz resonator
* transmission eval problems
* embedded evals.

Types of open problems

* Tomography / imaging / "inverse scattering"
* Bounds on physical properties of composite media
* What (besides symmetries) allows embedded eigenvalues?