Spectral decomposition of self-adjoint operators:
Motivation via finite-dimensional case.

$\mathcal{H}$: cx Hilbert space w/ inner product $(\cdot,\cdot)$
Assume $\dim(\mathcal{H}) < \infty$
Operator $A: \mathcal{H} \to \mathcal{H}$; The adjoint of A, A^*, satisfies $(Av,w) = (v, A^*w)$ $\forall\, v,w \in \mathcal{H}$
A self-adjoint means $A = A^*$, or $(Av,w) = (v,Aw)$.

From finite-dim'l linear algebra for self-adjoint A:

- A has real eigenvalues $\lambda_1, \dots, \lambda_m$ w/ eigenspaces $V_1, \dots, V_m$, with $V_i \perp V_j$ for $i \neq j$. $V_j = \mathrm{Null}(A - \lambda_j E)$
- Thus $\mathcal{H} = V_1 \oplus \dots \oplus V_m$

  $\mathcal{H} \ni v = v_1 + \dots + v_m$ unique decomposition (or resolution) of v

  $Av = \lambda_1 v_1 + \dots + \lambda_m v_m$ — the diagonalization of A.
- P_i = orthogonal projection onto V_i, i.e.

  i.e. $\mathrm{Im}\, P_i = V_i$ and $\mathrm{Null}\, P_i = \bigoplus_{j \neq i} V_j$

  i.e. $P_i P_j = \delta_{ij} P_i$

  Note P_i orthogonal projection iff P_i self-adjoint projection.
- $E = \sum_{i=1}^{m} P_i \leftarrow$ resolution of the identity operator E

  $A = \sum_{i=1}^{m} \lambda_i P_i \leftarrow$ spectral resolution of A (diagonalization of A)

- Functional calculus:

  $A^r = \sum \lambda_i^r P_i$

  $f(A) := \sum f(\lambda_i) P_i$
- Resolvent of A: $(A - zE)^{-1} = \sum_{i=1}^{m} \frac{1}{\lambda_i - z} P_i$

  Note: $\left((A-zE)^{-1} v, v\right) = \sum_{i=1}^{m} \frac{(P_i v, v)}{\lambda_i - z}$ takes the upper-half cx plane to itself! $\leftarrow$!

- Integral representation of the spectral decomposition of E (or $\mathcal{H}$):

$$E_\lambda := \sum_{j:\lambda_j \le \lambda} P_j \qquad (\text{for } \lambda \in \mathbb{R})$$

So $E_\lambda = \int_{-\infty}^{\lambda} dE_\mu$ (Riemann-Stieltjes integral)

$$E = \int_{-\infty}^{\infty} dE_\mu = \sum_{j=1}^{m} P_j$$

$$\cancel{E}v = Ev = \int dE_\mu v \; ; \qquad E_\lambda v = \int_{-\infty}^{\lambda} dE_\mu v$$

$$(v,w) = (Ev,w) = \int d(E_\mu v, w)$$

- Integral representation of the spectral decomposition of A:

$$A = \int \lambda \, dE_\lambda$$

$$Av = \int \lambda \, dE_\lambda v$$

$$f(A)v = \int f(\lambda) \, dE_\lambda v$$

Resolvent of A: $(A - zE)^{-1} = \int \frac{1}{\lambda - z} dE_\lambda$

Note that $\left((A-zE)^{-1} v, v \right) = \int \frac{1}{\lambda - z} d(E_\lambda v, v)$
(as a fcn of z) takes the upper half-plane to itself. This is because $(E_\lambda v, v)$ is an increasing fcn of λ.

⊙ Key of this motivation:
For infinite-dimensional Hilbert spaces, one has to extend E_λ to <u>increasing</u> <u>projection-valued</u> <u>functions</u>.