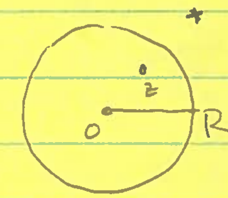
Toward the <u>Nevanlinna Theorem</u> for complex analytic functions from the upper-half plane to itself. (Thm. 3, p. 10)

Let $f(z)$ be a complex-analytic function defined in an open set that contains the closure $\overline{D}_R$ of the disk $D_R = \{z : |z| < R\}$ of radius R about 0. For $z \in \mathbb{C}$ with $|z| < R$, the Cauchy integral formula gives



$$f(z) = \frac{1}{2\pi i} \int_{C_R} \frac{f(\zeta)}{\zeta - z}\, d\zeta ,$$

where $C_R = \{z : |z| = R\}$ is the circle of radius R.

Define the <u>reflection z^* of z about C_R</u> by

$$z^* = R^2 \bar{z}^{-1} .$$

Since $|z^*| > R$, $f(\zeta)/(\zeta - z^*)$ is analytic in a neighborhood of D_R, and we obtain

$$0 = \frac{1}{2\pi i} \int_{C_R} \frac{f(\zeta)}{\zeta - z^*}\, d\zeta .$$

Substracting the two integrals yields

$$f(z) = \frac{1}{2\pi i} \int_{C_R} f(\zeta) \left(\frac{1}{\zeta - z} - \frac{1}{\zeta - z^*} \right) d\zeta .$$

Let us put $\zeta = Re^{it}$ and $z = \rho e^{i\phi}$. So $d\zeta = iRe^{it} dt$.
$= i\zeta dt$

This parameterization gives, for $\rho < R$,

$$f(\rho e^{i\phi}) = \frac{1}{2\pi}\int_0^{2\pi} f(Re^{it})\left(\frac{Re^{it}}{Re^{it}-\rho e^{i\phi}} - \frac{e^{it}}{e^{it}-R\rho^{-1}e^{i\phi}}\right)dt$$

$$= \frac{1}{2\pi}\int_0^{2\pi} f(Re^{it})\left(\frac{R^2-\rho^2}{R^2+\rho^2-2R\rho\cos(\phi-t)}\right)dt$$

The integral kernel in this expression is known as the Poisson kernel for the disk D_R :

$$K(R,t;\rho,\phi) = \frac{1}{2\pi}\frac{R^2-\rho^2}{R^2+\rho^2-2R\rho\cos(\phi-t)}$$

$$= \frac{1}{2\pi}\frac{|\zeta|^2-|z|^2}{|\zeta-z|^2}.$$

Notice that $K(R,t;\rho,\phi)$ is real-valued, and for $\rho < R$ it is positive. Thus one obtains integral representations for the real and imaginary parts of f.
If $f(z) = u(z) + iv(z)$,

$$u(\rho e^{i\phi}) = \int_0^{2\pi} u(Re^{it})\,K(R,t;\rho,\phi)\,dt$$

$$v(\rho e^{i\phi}) = \int_0^{2\pi} v(Re^{it})\,K(R,t;\rho,\phi)\,dt$$

The Poisson kernel produces a harmonic function (u or v) in the disk D_R in terms of its boundary values on the circle $C_R = \partial D_R$.

Now notice that the Poisson kernel is the real part of an analytic kernel:

(†) $$\frac{\zeta+z}{\zeta-z} = \frac{(\zeta+z)(\bar\zeta-\bar z)}{|\zeta-z|^2} = \frac{|\zeta|^2-|z|^2+2i\,\mathrm{Im}\, z\bar\zeta}{|\zeta-z|^2}$$

$$= \frac{R^2-\rho^2}{R^2+\rho^2-2R\rho\cos(\phi-t)} + i\,\frac{2R\rho\sin(\phi-t)}{R^2+\rho^2-2R\rho\cos(\phi-t)}$$

$$= 2\pi K(R,t;\rho,\phi) + i\,2\pi L(R,t;\rho,\phi)$$

[Note: as $\rho \to R$, K approaches the delta-function, or the identity as a convolution operator; and L as a convolution operator approaches the Hilbert transform on the circle.]

The function $g(z) = \frac{1}{2\pi}\int_0^{2\pi} u(Re^{it})\,\frac{Re^{it}+z}{Re^{it}-z}\,dt$

$$= \int_0^{2\pi} u(Re^{it})\,K(R,t;z)\,dt + i\int_0^{2\pi} u(Re^{it})\,L(R,t;z)\,dt$$

is analytic at all z in the open disk D_R and its real part is equal to $u(z)$.
Also notice that $g(0) = \int_0^{2\pi} u(Re^{it})\,dt = u(0)$, which is real.
Since an analytic function is determined up to an additive imaginary constant by its real part, we obtain

(∗) $$\boxed{f(z) = i\beta + \frac{1}{2\pi}\int_0^{2\pi} u(Re^{it})\,\frac{Re^{it}+z}{Re^{it}-z}\,dt, \quad v(0)=\beta.}$$

Source: Akhiezer & Glazman Ch. VI

Representation Theorems for analytic functions

Denote by D the open unit disk: $D = \{z : |z| < 1\}$; and by H_+ the open upper half plane: $H_+ = \{z : \operatorname{Im}(z) > 0\}$.

Theorem 1 A function $f : D \to \mathbb{C}$ is analytic and has a nonnegative real part if and only if there exists a real number β and an increasing function $\sigma : [0, 2\pi] \to \mathbb{R}$ such that, $\forall z \in D$,

$$f(z) = i\beta + \int_0^{2\pi} \frac{e^{it} + z}{e^{it} - z}\, d\sigma(t).$$

* Note $\beta = \operatorname{Im} f(0)$

Theorem 2 A function $f : H_+ \to \mathbb{C}$ is analytic and has a nonnegative imaginary part if and only if there exist real numbers α and $\mu \geq 0$ and an increasing function $\sigma : \mathbb{R} \to \mathbb{R}$ such that, $\forall z \in H_+$,

$$f(z) = \alpha + \mu z + \int_{-\infty}^{\infty} \frac{1 + tz}{t - z}\, d\sigma(t).$$

Theorem 3 A function $f : H_+ \to \mathbb{C}$ is analytic, has a nonnegative imaginary part, and satisfies

$$\limsup_{y \to \infty} |y f(iy)| < \infty$$

if and only if $\exists$ an increasing function $\sigma : \mathbb{R} \to \mathbb{R}$ of BV s.th. $\forall z \in H_+$,

$$f(z) = \int_{-\infty}^{\infty} \frac{1}{t - z}\, d\sigma(t).$$

Proof of Theorem 1 Assuming the given representation of f, and setting $\Phi(t,z) = (e^{it}+z)/(e^{it}-z)$, we have

$$\frac{f(z+\Delta z)-f(z)}{\Delta z} = \int_0^{2\pi} \frac{\Phi(t,z+\Delta z)-\Phi(t,z)}{\Delta z}\,d\sigma(t)$$

$$\longrightarrow \int_0^{2\pi} \frac{\partial\Phi}{\partial z}(t,z)\,d\sigma(t) \quad \text{as } \Delta z \to 0.$$

The convergence is valid by the following reasoning:
The difference quotient of Φ is continuous in t and Δz for $(t,\Delta z) \in [0,2\pi] \times \{\Delta z : |\Delta z| \le \varepsilon\}$ for some $\varepsilon > 0$, and thus Φ is uniformly continuous on this compact set.
Thus one obtains uniform convergence of the difference quotients to $\partial\Phi/\partial z\,(t,z)$ as $\Delta z \to 0$, and the convergence of the integrals follows. Thus f is analytic.
Recall from (+) p.9 that $\operatorname{Re}\Phi(t,z) > 0$. Given that σ is increasing, we find that $\operatorname{Re} f(z) \ge 0$.

Now assume $f : D \to \mathbb{C}$ is analytic and that $\operatorname{Re} f(z) \ge 0$ $\forall z \in D$. By the representation $(*)$ p.9, we have for $|z| < R < 1$,

$$(\circ) \qquad f(z) = \int_0^{2\pi} \frac{Re^{it}+z}{Re^{it}-z}\,d\sigma_R(t) + i\operatorname{Im} f(0)$$

in which $\sigma_R(t) = \dfrac{1}{2\pi}\displaystyle\int_0^t \operatorname{Re} f(Re^{is})\,ds \quad \forall\, t \in [0,2\pi]$.

We also have $0 \le \sigma_R(t) \le \dfrac{1}{2\pi}\displaystyle\int_0^{2\pi} \operatorname{Re} f(Re^{is})\,ds = \operatorname{Re} f(0)$.

Note: The Helly convergence theorem can be generalized by replacing f with a uniformly convergent sequence $f_n \to f \Rightarrow \int_a^b f_n \, d\sigma_n \to \int_a^b f \, d\sigma$

Since $\operatorname{Re} f(z) \geq 0$, σ_R is increasing on $[0, 2\pi]$, and the Helly selection theorem provides a sequence $\{R_j\}_{j=1}^{\infty}$ with $R_j \to 1$ as $j \to \infty$ and an increasing function $\sigma : [0, 2\pi] \to \mathbb{R}$ such that $\forall\, t \in [0, 2\pi]$,

$$\lim_{j \to \infty} \sigma_{R_j}(t) = \sigma(t).$$

From the Helly convergence theorem and (*) p. 11, we obtain

$$f(z) = i \operatorname{Im} f(0) + \int_0^{2\pi} \frac{e^{it} + z}{e^{it} - z} \, d\sigma(t). \quad \blacksquare$$

<u>Proof of Theorem 2</u> The given representation has nonnegative imaginary part whenever $\operatorname{Im} z \geq 0$ because

$$\operatorname{Im} \frac{1 + tz}{t - z} = \frac{1 + t^2}{|t - z|^2} \operatorname{Im} z \geq 0.$$

Analyticity will be shown later.

Let $f : H_+ \to \mathbb{C}$ be analytic with $\operatorname{Im} f(z) \geq 0$ $\forall\, z \in H_+$. Define a function $g : D \to \mathbb{C}$ by

$$g(\zeta) := -i f\left(i \frac{1+\zeta}{1-\zeta}\right) \quad \forall\, \zeta \in D.$$

This is well defined because the map $\zeta \mapsto i\frac{1+\zeta}{1-\zeta} = z$ takes D onto H_+. Also, g is analytic and $\forall\, \zeta \in D$, $\operatorname{Re} g(\zeta) = \operatorname{Im} f(z) \geq 0$ $\left(z = i(1+\zeta)/(1-\zeta)\right)$. By Theorem 1, there is an increasing function $\rho : [0, 2\pi] \to \mathbb{R}$ and a real number β such that $\forall\, \zeta \in D$,

$$(*) \qquad g(\varsigma) = i\beta + \int_0^{2\pi} \frac{e^{is}+\varsigma}{e^{is}-\varsigma}\, d\rho(s)$$

$$= i\beta + \frac{1+\varsigma}{1-\varsigma}\mu + \int_{0+0}^{2\pi-0} \frac{e^{is}+\varsigma}{e^{is}-\varsigma}\, d\rho(s),$$

in which $\int_{0-0}^{2\pi-0} \cdot\, d\rho(s)$ means $\int_{(0,2\pi)} d\mu_\rho$ (Lebesgue-Stieltjes integral)

and $\mu = \left(\rho(2\pi) - \rho(2\pi-0) + \rho(0+0) - \rho(0)\right)$.

For each $z \in H_+$, $\exists\ \varsigma \in D$ s.th. $z = i\frac{1+\varsigma}{1-\varsigma}$, namely $\varsigma = \frac{z-i}{z+i}$,

so $f(z) = ig(\varsigma) = -\beta + \mu z + \int_{0+0}^{2\pi-0} \frac{z\cot\frac{s}{2} - 1}{\cot\frac{s}{2} + z}\, d\rho(s)$.

Putting $\alpha = -\beta$, $t = -\cot\frac{s}{2}$ $(t \in \mathbb{R})$, and $\sigma(t) = \rho(2\operatorname{arccot}(-t))$, this becomes

$$f(z) = \alpha + \mu z + \int_{-\infty}^{\infty} \frac{1+zt}{t-z}\, d\sigma(t) \quad .$$

Finally, given any representation of this form, the transformation $\varsigma = \frac{z-i}{z+i}$ taking H_+ to D produces a function $g(\varsigma) = f(z)$ admitting the representation (*) p.13. Since g is analytic, so is f. ∎

<u>Proof of Theorem 3</u> Given σ as in the theorem, the function f defined by the integral has $\Im f(z) \geq 0$ $\forall\ z \in H_+$ since the integrand has positive imaginary part. The analyticity can be established by showing uniform (in t) convergence of the difference quotients of the integrand with respect to z; this is left to the reader.

Now suppose that $f : H_+ \to \mathbb{C}$ is analytic, $\operatorname{Im} f \geq 0$, and $\limsup\limits_{y\to\infty} |y f(iy)| < \infty$. By Theorem 2, there exists an increasing function $\rho : \mathbb{R} \to \mathbb{R}$ and real numbers α and μ such that $\forall\ z \in H_+$,

$$f(z) = \alpha + \mu z + \int_{-\infty}^{\infty} \frac{1+tz}{t-z}\, d\rho(t).$$

By the assumption on $yf(iy)$, we obtain, for some M,

$$\left| \alpha y + i\mu y^2 + \int_{-\infty}^{\infty} \frac{y(1+ity)}{t-iy}\, d\rho(t) \right| \leq M \quad \forall\ y \geq 0,$$

and taking real and imaginary parts yields

$$\left.\begin{aligned} &(1) \qquad y\left| \alpha + \int_{-\infty}^{\infty} \frac{(1-y^2)t}{t^2+y^2}\, d\rho(t) \right| \leq M \\ &(2) \qquad y^2\left| \mu + \int_{-\infty}^{\infty} \frac{1+t^2}{t^2+y^2}\, d\rho(t) \right| \leq M \end{aligned}\right\} \quad \forall\ y > 0$$

Inequality (2) yields $\mu = 0$, and (1) yields

Leb. dom. conv.: $|t|$ integrable and $\left|\frac{(y^2-1)t}{y^2+t^2}\right| < |t|$ for $y > 1$.

$$\alpha = \lim_{y\to\infty} \int_{-\infty}^{\infty} \frac{(y^2-1)t}{t^2+y^2}\, d\rho(t) = \int_{-\infty}^{\infty} t\, d\rho(t).$$

Thus

$$f(z) = \int_{-\infty}^{\infty} t\, d\rho(t) + \int_{-\infty}^{\infty} \frac{1+tz}{t-z}\, d\rho(t) = \int_{-\infty}^{\infty} \frac{1+t^2}{t-z}\, d\rho(t).$$

Inequality (2) implies $\displaystyle\int_{-\infty}^{\infty} y^2 \frac{1+t^2}{t^2+y^2}\, d\rho(t) \leq M \quad \forall\ y > 0$,

so that $\displaystyle\int_{-\infty}^{\infty} (1+t^2)\, d\rho(t) \leq M$ $\left(\text{take } b \to \infty \text{ in } \int_{-b}^{b}\right)$.

Because of this, the increasing function $\sigma : \mathbb{R} \to \mathbb{R}$ defined by

$$\sigma(t) = \int_{-\infty}^{t} (1+s^2)\, d\rho(s)$$

is of bounded variation, and $d\mu_\sigma(t) = (1+t^2)\, d\mu_\rho(t)$.

Finally,

$$f(z) = \int_{-\infty}^{\infty} \frac{1+t^2}{t-z}\, d\rho(t) = \int_{-\infty}^{\infty} \frac{1}{t-z}\, d\sigma(t). \qquad \blacksquare$$