Reference: Reed/Simon Vol 1, Ch. $\underline{VIII}$; see also Akhiezer & Glazman.

Unbounded operators in Hilbert space

An unbounded operator T in a Hilbert space $\mathcal{H}$ typically has a domain $\mathcal{D}(T) \subset \mathcal{H}$ that is dense in $\mathcal{H}$ but not equal to $\mathcal{H}$.

The graph of T, denoted by $\Gamma(T)$, is the relation in $\mathcal{H} \oplus \mathcal{H}$ associated with the operator T:

$$\mathcal{H} \oplus \mathcal{H} \supset \Gamma(T) = \{ \langle v, T(v) \rangle : v \in \mathcal{D}(T) \} .$$

Let $\overline{\Gamma(T)}$ be the closure of $\Gamma(T)$ in the Hilbert space $\mathcal{H} \oplus \mathcal{H}$ with inner product

$$\left(\langle v_1, v_2 \rangle \cdot \langle w_1, w_2 \rangle \right)_{\mathcal{H} \oplus \mathcal{H}} = (v_1, w_1)_{\mathcal{H}} + (v_2, w_2)_{\mathcal{H}} .$$

If $\overline{\Gamma(T)}$ is the graph of an operator, that is,
$\{ \langle v, w_1 \rangle \in \overline{\Gamma(T)}$ and $\langle v, w_2 \rangle \in \overline{\Gamma(T)} \} \Rightarrow w_1 = w_2$,
then T is said to be closeable, and its closure $\overline{T}$ is defined as the operator whose graph is $\overline{\Gamma(T)}$:

$$\Gamma(\overline{T}) = \overline{\Gamma(T)} \subset \mathcal{H} \oplus \mathcal{H} .$$

By definition, T is closed if $\Gamma(T) = \overline{\Gamma(T)}$, or, equivalently, if T is closeable and

$$T = \overline{T} .$$

An equivalent characterization of closedness is that T is closed if and only if
$$\{v_n \to v \text{ in } \mathcal{H},\ v_n \in \mathcal{D}(T) \text{ and } T(v_n) \to y \text{ in } \mathcal{H}\} \Longrightarrow \{v \in \mathcal{D}(T) \text{ and } T(v) = y\}.$$

The adjoint of T ($\mathcal{D}(T)$ assumed to be dense) is the operator T^* in $\mathcal{H}$ defined by

$$\begin{cases} \mathcal{D}(T^*) = \{ w \in \mathcal{H} : \mathcal{D}(T) \to \mathcal{H} :: v \mapsto (Tv, w) \text{ is bounded} \} \\ T^* w = w^*, \text{ where } w^* \text{ is the unique element} \\ \quad \text{of } \mathcal{H} \text{ s.th. } (Tv, w) = (v, w^*) \ \forall\, v \in \mathcal{D}(T). \end{cases}$$

Notice that such w^* exists because $v \mapsto (Tv, w)$ can be extended to a bounded linear functional on $\mathcal{H}$ in a unique way (BLT theorem) by the density of $\mathcal{D}(T)$, and by the Riesz representation theorem for Hilbert spaces.

Fact If $\mathcal{D}(T)$ is dense in $\mathcal{H}$, then T^* is closed, T is closeable, and $\overline{T} = T^{**}$.

Proof Define $V : \mathcal{H} \oplus \mathcal{H} \to \mathcal{H} \oplus \mathcal{H}$ by $V(\langle u, v\rangle) = \langle -v, u\rangle$. V is unitary and $V^2 = I$. Notice that, for $\langle w, w^*\rangle \in \mathcal{H} \oplus \mathcal{H}$,
$V(\langle v, T(v)\rangle) \cdot \langle w, w^*\rangle = \langle -T(v), v\rangle \cdot \langle w, w^*\rangle = -(T(v), w) + (v, w^*)$,
and this expression vanishes for all $v \in V$ if and only if $(w, w^*) \in \Gamma(T^*)$. This means that
$$\Gamma(T^*) = V[\Gamma(T)]^{\perp}.$$

Since perpendicular spaces are closed in Hilbert space, $\Gamma(T^*)$ is closed in $\mathcal{H} \oplus \mathcal{H}$, so T^* is a closed operator.
Next,

$$\begin{aligned}\Gamma(T^{**}) &= V[\Gamma(T^*)]^\perp = V[V[\Gamma(T)]^\perp]^\perp \\ &= V[V[\Gamma(T)]^{\perp\perp}] \quad \text{since } V \text{ is unitary} \\ &= V[\overline{V(\Gamma(T))}] = V[V(\overline{\Gamma(T)})] \\ &= V^2(\overline{\Gamma(T)}) = \overline{\Gamma(T)}\end{aligned}$$

This proves that $\overline{\Gamma(T)}$ is the graph of the operator T^{**}, and so T is closeable with $\overline{T} = T^{**}$. ∎

<u>Symmetric and self-adjoint operators</u>

By defn, T is <u>symmetric</u> if $(Tv, w) = (v, Tw) \quad \forall\ v, w \in \mathcal{D}(T)$.
Thus, in general,

$$\mathcal{D}(T) \subset \mathcal{D}(T^*) .$$

! → By defn., T is <u>self-adjoint</u> if $\underline{T = T^*}$, that is, <u>if T is symmetric and $\mathcal{D}(T) = \mathcal{D}(T^*)$</u>

There is a (very nice) theory on self-adjoint extensions of symmetric operators, and it has deep connections to physics.

The <u>spectrum</u> of T is the complement of the <u>resolvent set</u>

resolvent set $\longrightarrow$ $\rho(T) = \{ z \in \mathbb{C} : T - z$ has a bounded inverse $(T-z)^{-1} : \mathcal{H} \to \mathcal{D}(T) \}$,

spectrum $\longrightarrow$ $\sigma(T) = \mathbb{C} \setminus \rho(T)$.

<u>Theorem</u> If $T = T^*$, then $\sigma(T) \subset \mathbb{R}$.

<u>Proof</u> Let $z = x + iy \in \mathbb{C}$ be such that $y \neq 0$. For each $v \in \mathcal{D}(T)$,

$$
\begin{aligned}
\|(T-z)v\|^2 &= ((T-z)v, (T-z)v) \\
&= ((T-x)v, (T-x)v) + y^2(v,v) + \\
&\qquad - iy(Tv, v) + iy(v, Tv) \\
&= \|(T-x)v\|^2 + y^2\|v\|^2 \\
&\geq y^2\|v\|^2 \qquad (y^2 > 0)
\end{aligned}
$$

This shows that $\ker(T-z) = 0$ so that $T-z$ is injective. Similarly, $T - \bar{z}$ is injective.
To show that $\operatorname{Ran}(T-z)$ is dense, suppose that $w \in \mathcal{H}$ is such that $((T-z)v, w) = 0 \ \forall\, v \in \mathcal{D}(T)$. Thus $w \in \mathcal{D}(T^*) = \mathcal{D}(T)$, and since $T = T^*$,

$$0 = ((T-z)v, w) = (v, (T-\bar{z})w) \quad \forall\, v \in \mathcal{D}(T).$$

Since $\mathcal{D}(T)$ is dense in $\mathcal{H}$, we have $(T-\bar{z})w = 0$, and by the injectivity of $T - \bar{z}$, we find that $w = 0$. This proves that $(\operatorname{Ran}(T-z))^\perp = \{0\}$, so that $\operatorname{Ran}(T-z)$ is dense in $\mathcal{H}$.

The inequality $\|(T-z)v\| \geq |y|\|v\|$ shows that the inverse of $(T-z)$ on its range, $(T-z)^{-1} : \mathrm{Ran}(T-z) \to \mathcal{H}$ is bounded. By the BLT (bdd lin. transf.) Theorem, the closure of $(T-z)^{-1}$ is an operator whose domain is $\mathcal{H}$. But $(T-z)^{-1}$ is closed since T is closed, so $\mathcal{D}((T-z)^{-1}) = \mathcal{H}$, that is, $\mathrm{Ran}(T-z) = \mathcal{H}$.

Finally, since $T-z$ has a bounded inverse defined on $\mathcal{H}$ for each $z \in \mathbb{C}\setminus\mathbb{R}$, we have $\sigma(T) \subset \mathbb{R}$. ■

<u>Fact</u> A symmetric operator T is self-adjoint if and only if $\sigma(T) \subset \mathbb{R}$. If $\mathcal{D}(T) \not\subset \mathcal{D}(T^*)$, then $\sigma(T)$ contains the UHP or the LHP (or both).

<u>Fact</u> $\sigma(T^*) = \sigma(T)^*$.

<u>Example</u> Define $T_0 : C_c^\infty(0,1) \to L_2[0,1]$ by

$$(T_0 f)(x) = f''(x),$$

where $C_c^\infty[0,1]$ is the space of C^∞ functions with compact support in $(0,1)$. Such functions vanish in an open set about $\{0,1\}$ in $[0,1]$.

$\mathcal{D}(T_0^*) = H^2[0,1] = \{ f \in L^2[0,1] :$ the distributional derivative of f is in $L^2[0,1] \}$.

This is the Sobolev space of functions possessing a first and second derivative in the weak sense, with f, f', f'' all having finite mean square on $[0,1]$.

$\mathcal{D}(\bar{T}_0) = \mathcal{D}(T_0^{**}) = \{ f \in H^2[0,1] : f(0) = f'(0) = f(1) = f'(1) = 0 \}$.
(Values of f and f' make sense at 0 and 1 for $f \in H^2[0,1]$.)

Notice that, for each $z \in \mathbb{C}$, the set of functions

$$\{ t \mapsto e^{\sqrt{z}t}, \ t \mapsto e^{-\sqrt{z}t} \}$$

forms a basis for $\ker(T_0^* - z) = \ker(\partial_{xx} - z)$.
This is because $T_0^* f = \partial_{xx} f \ \forall f \in \mathcal{D}(T_0^*)$.
One can see this by integration by parts:
For $g \in \mathcal{D}(T_0)$ and $f \in \mathcal{D}(T_0^*)$,

$$(T_0 g, f) = \int_0^1 g''(t) f(t)\, dt = -\int_0^1 g'(t) f'(t)\, dt + g'(1)f(1) - g'(0)f(0)$$

$$= -\int_0^1 g'(t) f'(t)\, dt = \int_0^1 g(t) f''(t)\, dt - g(1)f'(1) + g(0)f'(0)$$

$$= \int_0^1 g(t) f''(t)\, dt = (g, \partial_{xx} f),$$

so $T_0^* = \partial_{xx}$ on $\mathcal{D}(T_0^*)$.

Now define T in $L^2[0,1]$ by

$$\begin{cases} \mathcal{D}(T) = \{ f \in L^2[0,1] : f(0) = 0,\ f(1) = 0 \}, \\ (Tf)(x) = f''(x) \quad \forall\ f \in \mathcal{D}(T). \end{cases}$$

It turns out that $T = T^*$.

T is a self-adjoint extension of T_0, and

(†) $$\mathcal{D}(T_0) \subsetneq \mathcal{D}(T) = \mathcal{D}(T^*) \subsetneq \mathcal{D}(T_0^*)$$

The theory of self-adjoint extensions of symmetric operators reveals that the set of self-adjoint extensions T of T_0 satisfy (†) and this set is identified with the 4-real-dimensional unitary group $U(2, \mathbb{C})$.