Concrete realizations of the spectral theorem: Fourier (harmonic) analysis

- "Recall" the Fourier transform on $L^2(\mathbb{R}^n)$.

$\hat{}: f \mapsto \hat{f}$, $\boxed{\hat{f}(\xi) = \int f(y) e^{-2\pi i \xi y} dy}$

is a unitary transformation from $L^2(\mathbb{R}^n)$ to $L^2(\mathbb{R}^n)$.
A concise basic treatment can be found in Chapter 0 of G. Folland's PDE book.

The Fourier inversion theorem

$$f(x) = \int \hat{f}(\xi) e^{2\pi i \xi x} dx$$

shows how an L^2 function can be written as an integral superposition of non-L^2 oscillatory functions $\chi_\xi(x) = e^{2\pi i \xi x}$.

- "Recall" also that if two diagonalizable linear transformations $A, B : \mathcal{H} \to \mathcal{H}$ in a finite-dimensional Hilbert space $\mathcal{H}$ commute, ie $AB = BA$, then they are simultaneously diagonalizable. Thus there are orthogonal projections P_j, $j = 1 \ldots m$, such that

$$A = \sum_{j=1}^{m} \lambda_j P_j \quad \text{and} \quad B = \sum_{j=1}^{m} \mu_j P_j$$

In terms of the resolution $E_t = \sum_{j: \lambda_j < t} P_j$, one has

$$A = \int t \, dE_t \quad \text{and} \quad B = \int m(t) \, dE_t$$

, so B is a fcn of A.

A view of Fourier analysis through translational symmetry.

The group $\mathbb{R}^n$ acts by translational symmetry on fcns on $\mathbb{R}^n$ as follows: For $y \in \mathbb{R}^n$, the associated translation operator S_y is defined by

$$(S_y f)(x) = f(x+y).$$

Here $x = (x_1 \dots x_n)$ and $y = (y_1 \dots y_n) \in \mathbb{R}^n$.

The "characters" of the group $\mathbb{R}^n$ are the homomorphisms from $\mathbb{R}^n$ into $\mathbb{C}$. The set of characters is a group also isomorphic to $\mathbb{R}^n$: For each $\xi \in \mathbb{R}^n$, the character $\chi_\xi : \mathbb{R}^n \to \mathbb{C}$ is

$$\chi_\xi : \mathbb{R}^n \to \mathbb{C} :: x \mapsto e^{2\pi i \xi x},$$

where $\xi x = \xi_1 x_1 + \dots + \xi_n x_n$. For each character χ_ξ, $\xi \in \mathbb{R}^n$, define the function $x \mapsto \hat{f}(x, \xi)$ by

$$\hat{f}(\cdot, \xi) = \int S_y f \, \chi_\xi(-y)\, dy,$$

whenever this integral makes sense (it can also be formal). This makes sense when $f \in L^2$ and $|\xi| = 1$ by a continuity argument (see Folland, e.g.)

One computes that

$$S_x \hat{f}(\cdot, \xi) = \int S_{x+y} f \, \chi_\xi(-y)\, dy = \int S_y f \; S_x \chi_\xi(-y)\, dy$$
$$= \int S_y f \, \chi_\xi(x-y)\, dy = \chi_\xi(x) \int S_y f \, \chi_\xi(-y)\, dy = \chi_\xi(x) \hat{f}(\cdot, \xi)$$

This means that $\hat{f}(\cdot, \xi)$ is an eigenfunction of the shifts S_x with eigenvalues $\chi_\xi(x)$. By evaluating at $\cdot = 0$, the equation $S_z \hat{f}(\cdot, \xi) = \chi_\xi(z) \hat{f}(\cdot, \xi)$ becomes

$$\hat{f}(x, \xi) = \chi_\xi(x) \hat{f}(0, \xi),$$

so that $\hat{f}(x, \xi)$ is just a multiple of $\chi_\xi(x)$, (now considered as a function of the underlying Euclidean space $\mathbb{R}^n$ rather than the group $\mathbb{R}^n$).

Many human beings are more accustomed to seeing this argument as follows:

$$\begin{aligned}\hat{f}(x, \xi) &= \int f(x+y) e^{-2\pi i \xi y} \, dy \\ &= \int f(y) e^{-2\pi i \xi (y-x)} \, dy \\ &= e^{2\pi i \xi x} \int f(y) e^{-2\pi i \xi y} \, dy \\ &= e^{2\pi i \xi x} \hat{f}(0, \xi),\end{aligned}$$

and notice that $\hat{f}(0, \xi)$ is the usual Fourier transform.

The Fourier transform converts the shift operators and any operator that commutes with the $\mathbb{Z}^n$ action into a multiplication operator. This is accomplished through Fourier inversion $f(x) = \int \hat{f}(x, \xi) \, d\xi$ because the functions $\hat{f}(x, \xi)$ are eigenfunctions of the $\mathbb{Z}^n$ action.

View the characters $\chi_\xi : \mathbb{R}^n \to \mathbb{C}$ as simultaneous eigenfunctions of the shifts S_y and any other operator commuting with the shifts.

Here is how different operators commuting with S_y are represented under the Fourier transform:

$$(S_y f)(x) = \int e^{2\pi i \xi y} \hat{f}(x,\xi)\,d\xi \qquad \text{shift}$$

$$-i\frac{\partial f}{\partial x_j}(x) = \int \xi_j \hat{f}(x,\xi)\,d\xi \qquad \text{derivatives}$$

$$-\Delta f(x) = \int |\xi|^2 \hat{f}(x,\xi)\,d\xi \qquad \text{Laplacian}$$

$$P\left(-i\frac{\partial}{\partial x_1}, \dots, -i\frac{\partial}{\partial x_n}\right) = \int P(\xi_1, \dots \xi_n)\hat{f}(x,\xi)\,d\xi \qquad P \text{ a polynomial}$$

const-coeff. linear diff. operator

$$(Kf)(x) := \int f(x-y)k(y)\,dy = \int \hat{k}(\xi)\hat{f}(x,\xi)\,d\xi \qquad \text{convolution}$$

BTW, convolutions by distributions k are the most general functions of the shift operators.

The associated spectral resolution of the identity in 1D:

$$E_\xi f := \int_{-\infty}^{\xi} \hat{f}(\cdot,\eta)\,d\eta = \int_{-\infty}^{\infty} \underbrace{1_{(-\infty,\xi)}\hat{f}(\cdot,\eta)}_{\text{"truncate in the frequency domain"}}\,d\eta$$

This resolution diagonalizes operators that are multiplication operators in Fourier space:

$$G\psi = \int g(\xi)\,dE_\xi \psi = \int g(\xi)\hat{f}(\cdot,\xi)\,d\xi .$$