Symmetries of the circle and Fourier series.

Let C be the unit circle $\{z : |z| = 1\} \subset \mathbb{C}$.
Its group of symmetries is also the circle; we call it G to distinguish its function from that of C. (C is a rotationally symmetric object, or "space", and G is its symmetry group.). Let's carry out the philosophy explained above for $\mathbb{R}^n$, now for the circle:

- Space C = unit circle in $\mathbb{C}$, as a set (w/ Lebesgue measure)
- Symmetry group of C : G = unit circle in $\mathbb{C}$ as a group.

- Action of G on C is by multiplication, which effects a rotation of C by each element of G:
  For $\varsigma = e^{i\phi} \in G$ the action is
  $\varsigma : C \to C :: z \mapsto \varsigma z$, or $e^{i\theta} \mapsto e^{i(\theta+\phi)}$
  ↑ shift in exponent

- Induced action on $L^2(C)$:
  For $\varsigma = e^{i\phi} \in G$, define the rotation R_ς by
  $R_\varsigma : L^2(C) \to L^2(C)$
  $(R_\varsigma f)(z) = f(\varsigma z)$, ie $(R_{e^{i\phi}} f)(e^{i\theta}) = f(e^{i(\theta+\phi)})$

- Characters of G are continuous homomorphisms $G \to \mathbb{C}^*$
  They are of the form

$$\chi_n : G \to \mathbb{C}^* :: \varsigma \mapsto \varsigma^n, \quad \text{ie} \quad e^{i\phi} \mapsto e^{in\phi}$$

for $n \in \mathbb{Z}$. So the group of characters is isomorphic to $\mathbb{Z}$.

- Fourier transform in $L^2(C)$ with respect to the group G:

Given $f \in L^2(C)$, define $\hat{f} : C \times \mathbb{Z}$ by

$$\hat{f}(\cdot, n) = \int_G R_\zeta f \, \chi_n(\zeta^{-1}) \, d\zeta$$

i.e., $$\hat{f}(e^{i\theta}, n) = \int_0^{2\pi} f(e^{i(\theta+\phi)}) e^{-in\phi} \, d\phi$$

- Show that $\hat{f}(\cdot, n)$ is a simultaneous eigenfunction for the G action, meaning for each rotation R_ζ, $\zeta \in G$:

$$(R_{e^{i\phi'}} \hat{f})(e^{i\phi}, n) = \hat{f}(e^{i(\phi+\phi')}, n) = \int_0^{2\pi} f(e^{i(\theta+\phi+\phi')}) e^{-in\phi} \, d\phi$$

$$= \int_0^{2\pi} f(e^{i(\theta+\phi)}) e^{-in(\phi-\phi')} \, d\phi = e^{in\phi'} \int_0^{2\pi} f(e^{i(\theta+\phi)}) e^{-in\phi} \, d\phi$$

$$= e^{in\phi'} \hat{f}(e^{i\phi}, n), \quad \text{so} \quad \hat{f}(e^{i\theta}, n) = \hat{f}(1, n) e^{in\theta}$$

- Fourier inversion theorem

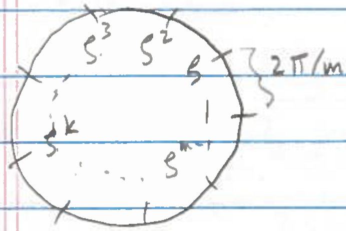
$$f = \sum_{n \in \mathbb{Z}} \hat{f}(\cdot, n)$$

ie $$f(e^{i\theta}) = \sum_{n \in \mathbb{Z}} \hat{f}(e^{i\theta}, n) = \sum_{n \in \mathbb{Z}} \hat{f}(1, n) e^{in\theta}$$

- Operators that commute with the G action (ie with all R_ζ):

- differential operators $a_n \frac{\partial^n}{\partial\theta^n} + \ldots + a_1 \frac{\partial}{\partial\theta} + a_0$
- convolution operators, such as the Hilbert transform (recall this is the imaginary part of the Poisson kernel restricted to C)

Finite symmetry group of the circle



Consider an operator L acting on functions defined on the unit circle C as follows:

$$L = -\partial_{\theta\theta} + q(\theta)$$

Here, $-\partial_{\theta\theta}$ is minus the second derivative with respect to the angular variable θ. Since this variable is in $\mathbb{R}/2\pi$, functions on C will be denoted by $f(\theta)$, where θ represents an element of $\mathbb{R}/2\pi$.
[Thus, $q(\theta + 2\pi n) = q(\theta)$ for all $n \in \mathbb{Z}$.]

Now suppose that $q(\theta)$ is invariant under rotating the circle C by $2\pi/m$, i.e.

$$q\left(\theta + \frac{2\pi}{m}\right) = q(\theta).$$

Thus $q\left(\theta + \frac{2\pi k}{m}\right) = q(\theta)$, so the operator L <u>commutes with</u> rotations by angles $2\pi k/m$.

<u>Exercise</u> – Show this formally, i.e., $L R_{\zeta^k} = R_{\zeta^k} L$, where $\zeta = e^{2\pi i/m}$.

So L commutes with the action of the cyclic group of order m on C,

$$\mathbb{Z}_m = \{R_{\zeta^k} : k = 0, \ldots, m-1\} \cong \mathbb{Z}/m\mathbb{Z}$$

Let's carry out the harmonic (Fourier) analysis for the group $\mathbb{Z}_m$ acting on the circle C.

- Space C = unit circle in $\mathbb{C}$, as a set
- A partial symmetry group of C: $\mathbb{Z}_m$ = cyclic grp of order m of rotations by angles $2\pi k/m$, $k = 0, \dots, m-1$.

- Action of $\mathbb{Z}_m$ on C is as before:
For $\zeta^k \in \mathbb{Z}_m$, $e^{i\theta} \mapsto \zeta^k e^{i\theta} = \exp i(\theta + 2\pi k/m)$.

- Induced action of $\mathbb{Z}_m$ on $L^2(C)$:
$R_k : L^2(C) \to L^2(C)$, $(R_k f)(\theta) = f(\theta + 2\pi k/m)$

- Characters of $\mathbb{Z}_m$ are homomorphisms $\mathbb{Z}_m \to \mathbb{C}^*$.
They are of the form

$$\chi_\ell : \mathbb{Z}_m \to \mathbb{C}^* :: \zeta^k \mapsto \zeta^{\ell k}, \qquad \ell = 0, \dots, m-1.$$

ie the generator ζ of $\mathbb{Z}_m$ is mapped by χ_ℓ to ζ^ℓ.
The group of characters is isomorphic to C_m.

- Fourier transform in $L^2(C)$ with respect to the $\mathbb{Z}_m$ action:

Given $f \in L^2(C)$, define $\hat{f} : C \times \mathbb{Z}_m$ by

$$\hat{f}(\cdot, \ell) = \sum_{k=0}^{m-1} R_k f \, \chi_\ell(\zeta^k)$$

In human terms, this is

$$\hat{f}(\theta, \ell) = \sum_{k=0}^{m-1} f(\theta + 2\pi k/m)\, e^{-2\pi i k\ell/m}$$

- Show that $\hat{f}(\theta,\ell)$ is an e-fcn of the $\mathbb{Z}_m$-action:

$$\begin{aligned}
(R_{k'}\hat{f})(\theta,\ell) &= \hat{f}(\theta + 2\pi k'/m, \ell) \\
&= \sum_{k=0}^{m-1} f(\theta + 2\pi(k+k')/m)\, e^{-2\pi i k\ell/m} \\
&= \sum_{k=0}^{m-1} f(\theta + 2\pi k/m)\, e^{-2\pi i (k-k')\ell/m} \\
&= e^{2\pi i k'\ell/m} \sum_{k=0}^{m-1} f(\theta + 2\pi k/m)\, e^{-2\pi i k\ell/m} \\
&= e^{2\pi i k'\ell/m}\, \hat{f}(\theta,\ell)
\end{aligned}$$

- This shows that $\hat{f}(\theta,\ell)$ is determined by its values on the segment of the circle corresponding to the angular interval $[0, 2\pi/m)$.

Given θ, one can write it uniquely as

$$\theta = \tilde{\theta} + 2\pi k/m, \quad \tilde{\theta} \in [0, 2\pi/m), \; k \in \mathbb{Z}_m$$

$$\hat{f}(\theta,\ell) = \hat{f}(\tilde{\theta} + 2\pi k/m, \ell) = \hat{f}(\tilde{\theta},\ell)\, e^{2\pi i k\ell/m}.$$

$\hat{f}(\tilde{\theta},\ell)$ is a function in $L^2([0,2\pi/m))$; it is the generalization of the Fourier coefficients from before.

[Note:
The reason that we now have coefficients $\hat{f}(\tilde{\theta}, l)$ that are in $L^2([0, 2\pi/m))$ instead of just being scalars is that the action of $\mathbb{Z}_m$ on C is not transitive — it does not resolve the interval $[0, 2\pi/m)$. This interval is a fundamental domain for the $\mathbb{Z}_m$ action on C.]

- Fourier inversion theorem

$$f = \sum_{l=0}^{m-1} \hat{f}(\cdot, l)$$

sum over characters

$$\text{ie,} \quad f(\theta) = \sum_{l=0}^{m-1} \hat{f}(\theta, l)$$

$$\text{ie,} \quad f(\tilde{\theta} + 2\pi k/m) = \sum_{l=0}^{m-1} \hat{f}(\tilde{\theta}, l)\, e^{2\pi i k l/m}$$

The "Fourier coefficients" are in L^2 of the fundamental domain $[0, 2\pi/m)$ of the $\mathbb{Z}_m$ action on C.