Floquet Theory

"Floquet theory" is Fourier theory with respect to ~~the~~ a faithful action of $\mathbb{Z}^n$. It is used to analyze the spectral properties of periodic differential operators, such as the Schrödinger operator $-\Delta + q(x)$ in $\mathbb{R}^n$, where $q(x)$ is a periodic potential, with n independent vectors of periodicity $v_1 \dots v_n$. This means that

$$q(x+v_j) = q(x) \quad \forall\, x \in \mathbb{R}^n,\ j = 1, \dots, n.$$

- $X = \mathbb{R}^n$ (underlying space)
- $G = \mathbb{Z}^n$ (group of translational symmetries of X).

with action as follows: For $g = (g_1 \cdots g_n) \in G$ and $x = (x_1 \dots x_n) \in X$,

$$gx = x + g_1 v_1 + \dots + g_n v_n = x + g\bar{v}$$

The induced action on functions is, as usual,

$$(gf)(x) = f(gx) = f(x + g_1 v_1 + \dots + g_n v_n).$$

- Each character χ of G is determined by the numbers $\chi(e_j) \in \mathbb{C}^*$, $j = 1 \dots n$, where $e_j = (0, \dots, 1, \dots 0)$.

Thus the characters are in one-to-one correspondence with $(\mathbb{C}^*)^n$. For each $z = (z_1 \dots z_n) \in (\mathbb{C}^*)^n$, $\chi_z(g) = z_1^{g_1} \cdots z_n^{g_n} =: z^g$.

For $z, w \in (\mathbb{C}^*)^n$, $\quad \chi_z(g)\,\chi_w(g) = z^g w^g = (zw)^g = \chi_{zw}(g)$, and thus the set of characters has a group structure isomorphic to that of $(\mathbb{C}^*)^n$. The unitary characters form a group isomorphic to the n-dimensional torus

$$\mathbb{T}_n = \{ z \in (\mathbb{C}^*)^n : |z_1| = \dots = |z_n| = 1 \}$$
$$= \{ z \in \mathbb{C}^* : |z| = 1 \}^n$$

It is called the "dual group" G^* to G : $G^* \cong \mathbb{T}_n$.

The "Floquet transform" of a function $f : X \to \mathbb{C}$ is

$$\hat{f}(x, z) = \sum_{g \in \mathbb{Z}^n} f(x + g\bar{v})\, z^{-g}$$

Notice that (again), $\hat{f}(\cdot, z)$ is an eigenfunction of the action of $g \in G$, with eigenvalue z^g:

$$g\hat{f}(\cdot, z) = \hat{f}(\cdot + g\bar{v}, z) = \sum_{h \in \mathbb{Z}^n} (\cdot + g\bar{v} + h\bar{v})\, z^{-h}$$
$$= \sum_{h \in \mathbb{Z}^n} f(\cdot + h\bar{v})\, z^{(g-h)} = z^g\, \hat{f}(\cdot, z).$$

Such a function on $\mathbb{R}^n$ is called "pseudo-periodic", or "quasi-periodic", with Floquet multipliers $(z_1, \dots, z_n)$. If ~~$z_1, \dots, z_n \in$ all~~ $z \in \mathbb{T}_n$, then the numbers $z_j = e^{ik_j}$ are associated to "phase shifts" k_j.

Let W be a fundamental domain in $\mathbb{R}^n$ of the G action. Specifically, let W be the parallelopiped determined by $v_1 \dots v_n$.

One has $\quad \hat{f}(x+g\bar{v}, z) = z^g \hat{f}(x,z)$.

Since each element of $\mathbb{R}^n$ is uniquely written as $x+g\bar{v}$ with $x \in W$ and $g \in \mathbb{Z}^n$, this equation shows that $\hat{f}(\cdot, z)$ as a fcn on $\mathbb{R}^n$ is determined by its restriction $\hat{f}(\cdot, z)|_W$ to W.

Restricting z to $\mathbb{T}_n$, one can consider $\hat{f}$ to be a function on $\mathbb{T}_n$ with values in $L^2(W)$.

We will use the notation $\hat{f}(x,k)$ in place of $\hat{f}(x, e^{ik})$ when we consider $k \in \underset{=: W^*}{[0, 2\pi)^n}$. [This will be considered a tolerable abuse of notation.]

W^* = dual fundamental domain

Let's show the following facts:

- $f \mapsto \hat{f}$ is a unitary isomorphism from $L^2(\mathbb{R}^n)$ to $L^2(\mathbb{T}_n, L^2(W))$.

- $$f(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{T}_n} \hat{f}(x,z)\, dz = \frac{1}{(2\pi)^n} \int_{W^*} \hat{f}(x,k)\, dk$$
$$= \frac{1}{(2\pi)^n} \int_0^{2\pi} \cdots \int_0^{2\pi} \hat{f}(x, k_1, \dots, k_n)\, dk_1 \dots dk_n$$

The first $(\cdot)$ can ~~bee~~ seen as follows:

(Pf) Using the L^2-preserving property of the discrete Fourier transform, one observes that

$$(*) \qquad \hat{f}(x,k) = \sum_{g \in \mathbb{Z}^n} f(x+g)\, e^{-ikg}$$

is a discrete F-transform for each $x \in W$, ~~Thus~~ and thus

$$\sum_{g \in \mathbb{Z}^n} |f(x+g)|^2 = \frac{1}{(2\pi)^n} \int_{\mathbb{T}_n} |\hat{f}(x,k)|^2\, dk$$

$$\|f\|_{L^2(\mathbb{R}^n)} = \int_{\mathbb{R}^n} |f(x)|^2\, dx = \int_W \sum_{g \in \mathbb{Z}^n} |f(x+g)|^2\, dx$$

$$= \frac{1}{(2\pi)^n} \int_{\mathbb{T}^n} \int_W |\hat{f}(x,k)|^2\, \cancel{dx}\, dk$$

$$= \frac{1}{(2\pi)^n} \int_{\mathbb{T}^n} \|\hat{f}(\cdot,k)\|_{L^2(W)}\, dk$$

$$= \|\hat{f}\|_{L^2(\mathbb{T}_n, L^2(W))} .$$

The second $(\cdot)$, Fourier inversion, also comes from inversion of the discrete Fourier transform: ~~(*)~~

$$\text{(Pf)} \quad f(x+g) \overset{(*)\Rightarrow}{=} \frac{1}{(2\pi)^n} \int_{W^*} \hat{f}(x,k)\, e^{ikg}\, dk = \frac{1}{(2\pi)^n} \int_{W^*} \hat{f}(x+g,k)\, dk$$

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Now let's ~~xx~~ see how the F-transform of f with respect to the full translational group $\mathbb{R}^n$ of $X = \mathbb{R}^n$ is a refinement of the F-transform of f with respect to the discrete subgroup $\mathbb{Z}^n \subset \mathbb{R}^n$ of translational symmetries of $\mathbb{R}^n$.

$$\mathcal{F}f(\xi) = \int_{\mathbb{R}^n} f(x)\, e^{-i\xi x}\, dx$$

$$= \sum_{g \in \mathbb{Z}^n} \int_{W+g} f(x)\, e^{-i\xi x}\, dx$$

$$= \sum_{g \in \mathbb{Z}^n} \int_{W} f(x+g)\, e^{-i\xi(x+g)}\, dx$$

Now set $\xi = 2\pi m + k$, $m \in \mathbb{Z}^n$ and $k \in W^* := [0, 2\pi)^n$

$$\mathcal{F}f(2\pi m + k) = \int_W \sum_{g \in \mathbb{Z}^n} f(x+g)\, e^{-i(2\pi m + k)(x+g)}\, dx$$

$$= \int_W \left[\left(\sum_{g \in \mathbb{Z}^n} f(x+g)\, e^{-ikg}\right) e^{-ikx}\right] e^{-2\pi i m x}\, dx$$

$$= \int_W \left[\hat{f}(x,k)\, e^{-ikx}\right] e^{-2\pi i m x}\, dx$$

$$= \int_W \underbrace{\tilde{\hat{f}}(x,k)}\, e^{-2\pi i m x}\, dx \quad \longleftarrow \;!!$$

$$:= \hat{f}(x,k)\, e^{-ikx} \;\longleftarrow\; \text{periodic "version" of pseudo-per. fcn.}$$