Remember that the purpose of the brand of harmonic analysis presented in these notes (i.e., generalized Fourier analysis) is to obtain a concrete realization of the abstract spectral theorem we have proved. Let's see just how this works.

Case $X = \mathbb{R}^n$, $G = \mathbb{R}^n$, $G^\vee = \mathbb{R}^n$

The F-transform of $f \in L^2(\mathbb{R}^n)$ is $\hat{f}(x,\xi) = \int_{\mathbb{R}^n} f(x+g)\, e^{-2\pi i \xi g}\, dg$

Recall that $\hat{f}(\cdot,\xi)$ is an eigenfunction of the group action of G with eigenvalue equal to the character $g \mapsto e^{2\pi i \xi g}$:

$$g\hat{f}(\cdot,\xi) = e^{2\pi i \xi g}\, \hat{f}(\cdot,\xi)$$

and since the G-action is transitive, $\hat{f}(x,\xi) = e^{2\pi i \xi x}\, \hat{f}(\xi)$, with $\hat{f}(\xi) := \hat{f}(0,\xi)$. The F-inversion theorem builds f as a superposition of these eigenfunctions:

$$f = \int_{\mathbb{R}^n} \hat{f}(\cdot,\xi)\, d\xi$$

Let's take $n=1$. In this case, one defines quite immediately the orthogonal projections E_ξ (for $\xi \in \mathbb{R}$):

$$E_\xi f = \int_{-\infty}^{\xi} \hat{f}(\cdot,\eta)\, d\eta \, . \qquad [\text{Concrete defn. of } E_\xi \,!]$$

It is easy to show that $\{E_\xi\}_{\xi\in\mathbb{R}}$ is a resolution of the identity (exercise). One has

$$f = \int_{-\infty}^{\infty} dE_\xi f$$

Operators that commute with the G-action (translation) become diagonalized under this resolution of $\mathbb{E}$:

The action of $g \in G$:

$$gf = \int_{-\infty}^{\infty} e^{2\pi i \xi g} \, dE_\xi f$$

Derivatives:

$$\frac{d}{dx} f = \int_{-\infty}^{\infty} 2\pi i \xi \, dE_\xi f$$ (use the expression for gf above)

$$\frac{1}{2\pi i} \frac{d}{dx} f = \int_{-\infty}^{\infty} \xi \, dE_\xi f$$ ← This is the canonical form we derived for a self-adj. operator.

$$\left(\frac{1}{i} \frac{d}{dx} - zE \right)^{-1} = \int_{-\infty}^{\infty} \frac{1}{2\pi \xi - z} \, dE_\xi$$ ← Resolvent of $\frac{1}{i}\frac{d}{dx}$ (functional calculus!)

$$-\frac{d^2}{dx^2} f = \int_{-\infty}^{\infty} (2\pi \xi)^2 \, dE_\xi f$$ – Laplacian in 1D

Convolution:

$$(Kf)(x) = \int k(x-y) f(y) \, dy$$

$$\Rightarrow Kf = \int \check{k}(\xi) \, dE_\xi f$$ (prove this) ← exercise

Let's see how we can use this to solve the homogeneous wave equation:

$$\frac{\partial^2}{\partial t^2} u(x,t) = c^2 \frac{\partial^2}{\partial x^2} u(x,t)$$

Let's write the F-transform (spectral decomposition) as explicitly as we can (assuming smoothness for now):

$$u(x,t) = \int \hat{u}(\xi,t) e^{2\pi i \xi x} \, d\xi$$

$$c^2 \frac{\partial^2 u}{\partial x^2}(x,t) = c^2 \int (2\pi \xi)^2 \hat{u}(\xi,t) e^{2\pi i \xi x} \, d\xi$$

$$\frac{\partial^2 u}{\partial t^2}(x,t) = \int \frac{d^2}{dt^2} \hat{u}(\xi,t) e^{2\pi i \xi x} \, d\xi$$

$$\Rightarrow \quad \frac{\partial^2 \hat{u}}{\partial t^2}(\xi,t) = (2\pi c \xi)^2 \hat{u}(\xi,t)$$

$$\Rightarrow \quad \hat{u}(\xi,t) = a(\xi) \cos 2\pi c \xi t + b(\xi) \sin 2\pi c \xi t$$

$$\Rightarrow \quad u(x,t) = \int \left(a(\xi) \cos 2\pi c \xi t + b(\xi) \sin 2\pi c \xi t \right) e^{2\pi i \xi x} \, d\xi$$

This kind of analysis goes through in $\mathbb{R}^n$, ie for the wave equation $\frac{\partial^2}{\partial t^2} u = -c^2 \Delta u$.

Suppose now that we have an inhomogeneous wave equation that, nonetheless, possesses a certain amount of homogeneity in that its coefficients are periodic, meaning that the wave operator commutes with discrete translations. We may have, for example,

$$\frac{\partial^2 u}{\partial t^2} = c(x)^2 \frac{\partial^2 u}{\partial x^2} \quad \left(\text{or } c(x)^2 \Delta u\right)$$

or $$\frac{\partial^2 u}{\partial t^2} = a(x) \frac{\partial}{\partial x}\left[b(x) \frac{\partial u}{\partial x} \right] \quad \left[\text{or } a(x)\, \nabla \cdot \left(b(x) \nabla u\right)\right]$$

or $$i \frac{\partial}{\partial t} u = -\Delta u + q(x) ,$$

...

in which the coefficients are invariant under a discrete translation group action of $\mathbb{Z}^n$ on $\mathbb{R}^n$.

Now we must use the <u>Floquet</u> transform.

Lets deal with the Schrödinger operator

$$-\Delta + q(x),$$

with $q(x+g\bar{v}) = q(x) \quad \forall\ g \in \mathbb{Z}^n \qquad (\bar{v} = [v_1, \dots, v_n]$ as before$)$.

Let's use the following definition of the Floquet transform of $f \in L^2(\mathbb{R}^n)$:

$$\hat{f}(x,k) = \frac{1}{(2\pi)^n} \sum_{g \in \mathbb{Z}^n} f(x+g\bar{v})\, e^{-ikg}$$

The inversion formula gives f as a superposition of these quasi-periodic functions $\hat{f}(x,k)$:

$$f(x) = \int_{\mathbb{T}_n} \hat{f}(x,k)\, dk$$

We would like to understand the <u>spectrum of $-\Delta + q$</u>, and this is accomplished ~~by~~ through the F-transform:

Consider the equation

$$(-\Delta + q(x) - E)\, u = f,$$

with $f \in L^2(\mathbb{R}^n)$. By applying the F-transform & inverse, one obtains

$$(-\Delta + q(x) - E) \int_{\mathbb{T}_n} \hat{u}(x,k)\, dk = \int_{\mathbb{T}_n} \hat{f}(x,k)\, dk$$

$$\Downarrow$$

$$= \int_{\mathbb{T}_n} (-\Delta + q(x) - E)\, \hat{u}(x,k)\, dk$$

From this, one obtains

$$\left(-\Delta + q(x) - E\right)\hat{u}(x,k) = \hat{f}(x,k) \quad \forall \quad k \in \mathbb{T}_n$$

The problem looks as if it depended ~~not~~ on k only through the "forcing" $\hat{f}(x,k)$. But in fact, the dependence on k is also in the left-hand side because $\hat{u}(x,k)$ is quasi-periodic and is locally in H^2 (ie $\hat{u}(\cdot,k) \in H^2_{loc}(\mathbb{R}^n)$). Thus, on the fundamental domain W (parallelopiped determined by the vectors $v_1 \ldots v_n$), the function $\hat{u}(\cdot,k)$ satisfies the boundary conditions

$$\hat{u}(x+v_j,k) = e^{ik_j}\hat{u}(x,k)$$

$$v_j \cdot \nabla\hat{u}(x+v_j,k) = e^{ik_j}\, v_j \cdot \nabla\hat{u}(x,k) \qquad \begin{matrix}(\text{for } x \in \partial W) \\ (\text{in partic.})\end{matrix}$$

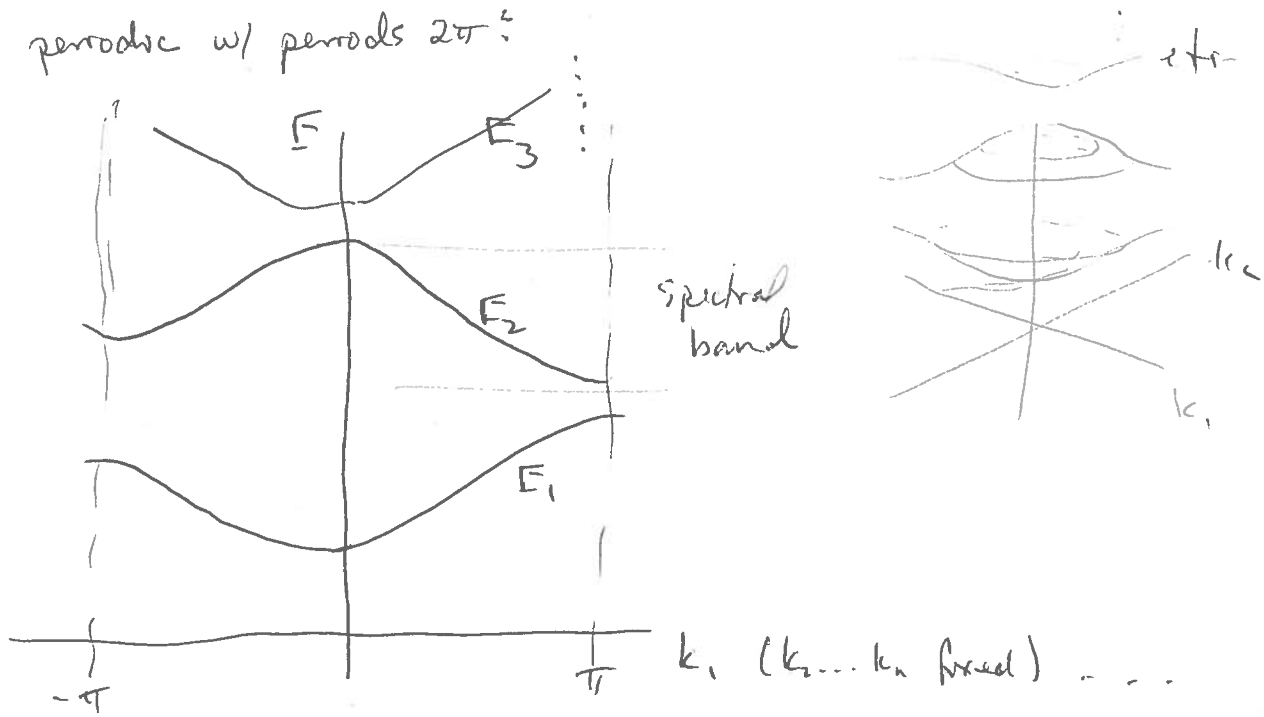
It turns out that these boundary conditions serve ~~as~~ to determine a domain $\mathcal{D}_k \subset H^2(W)$ on which $-\Delta + q(x)$ is a <u>self-adjoint</u> operator.

Whether E is a spectral value of $-\Delta + q(x)$ on $L^2(\mathbb{R}^n)$ depends on whether $-\Delta + q(x) - E$ is invertible in <u>each</u> of the domains $\mathcal{D}_k$.

It turns out that the spectrum of

$$A_k = -\Delta + q(x) - E \quad \text{on } \mathcal{D}_k$$

is discrete. Thus it consists of eigenvalues $E_j(k)$, $j = 1, 2, 3, \dots$ In the k-variable, these "band functions" are periodic w/ periods 2π:



See "An Overview of Periodic Elliptic Operators" by Peter Kuchment,

The spectrum of $-\Delta + q(x)$ in $L^2(\mathbb{R}^n)$ is all E such that A_k is not invertible, for some k, ie, the union of all of the $E_j(k)$: $\sigma(A) = \bigcup_{\substack{j=1,2,\dots \\ k \in T_n}} E_j(k)$. It can happen that the spectrum of A contains

$\boxed{\text{spectral gaps!}}$ ← big deal!! – google it!