A simple <u>waveguide with an obstacle</u>.

We will develop the spectral theory for the wave equation in an infinite string with a localized defect that acts as a scatterer of waves.

The main arguments will be outlined first, omitting technical lemmas in order to elucidate the essential ideas.

1. <u>Undistorted waves</u>. The wave equation in a uniform string

$$\partial_{tt} u = c^2 \partial_{xx} u$$

admits harmonic solutions of the form $u = e^{i(kx - \omega t)}$, where $\omega = ct$. The (normalized) spatial factor

$$w_0(x; k) = \frac{1}{\sqrt{2\pi}} e^{ikx}$$

is a <u>generalized eigenfunction</u> of the positive "propagator" $-c^2 \partial_{xx}$: $\quad -c^2 \partial_{xx} w_0 = (ck)^2 w_0$, or

ex
$$\left(-\partial_{xx} - k^2\right) w_0(x; k) = 0$$

The usual Fourier transform is based on these undisturbed wave functions. For $\psi \in L^2(\mathbb{R})$,

$$\hat{\psi}_0(k) := (\mathcal{F}\psi)(k) = \int \psi(x)\, \overline{w}_0(x; k)\, dx \ .$$

2. <u>Distorted waves</u> due to an obstacle in the waveguide.

Let the density and rigidity of the string be modified
in a finite region $[-L, L]$:

$$\rho \, \partial_{tt} u = c^2 \partial_x (\tau \partial_x u),$$

$$\begin{cases} 0 < \sigma_- \leq \sigma(x) = \dot{\rho}^{-1}(x) \leq \sigma_+, \\ \sigma(x) = 1 \quad \text{for } |x| > L, \end{cases}$$

(the obstacle in $[-L, L]$)

$$\begin{cases} 0 < \tau_- \leq \tau(x) \leq \tau_+, \\ \tau(x) = 1 \quad \text{for } |x| > L. \end{cases}$$

The new propagator is the $\overset{\text{positive}}{\text{operator}}$ $-\sigma (\partial_x \tau \partial_x)$.
We seek generalized eigenfunctions $w(x;k)$ that
modify the functions $w_0(x;k)$ in a prescribed way:

$$-\sigma (\tau w')' - k^2 w = 0$$

(†)
$$w(x;k) = w_0(x,k) + u(x;k) \qquad \text{(distorted waves)}$$

$$u(x;k) = \begin{cases} u_+ e^{i|k|x} & \text{for } x > L \\ u_- e^{-i|k|x} & \text{for } x < -L \end{cases} \qquad \text{(radiating condition)}$$

These wave functions $w(x;k)$ are called <u>distorted wave functions</u>
or <u>scattering fields</u> because an "incident field"
$w_0(x;k)e^{-i\omega t}$ is scattered by the obstacle, producing
a diffracted field that <u>radiates</u> away from the obstacle
to $\pm\infty$. We take $\omega = c|k|$ in the time dynamics.

For $k > 0$, $w(x, k)$ is interpreted as the field produced by the scattering by the obstacle of a harmonic wave produced by an oscillatory source at $x = -\infty$; and for $k < 0$, $w(x, k)$ represents scattering of a field emanating from the right:

$k > 0$:
$$\sqrt{2\pi}\, w(x; k) = \begin{cases} \left(e^{i|k|x} + r e^{-i|k|x} \right) e^{-i\omega t} , & x < -L \\ t e^{i|k|x} e^{-i\omega t} , & x > L \end{cases}$$

$\omega = c|k| > 0$

$k < 0$:
$$\sqrt{2\pi}\, w(x; k) = \begin{cases} t e^{-i|k|x} e^{-i\omega t} , & x < -L \\ \left(e^{-i|k|x} + r e^{i|k|x} \right) e^{-i\omega t} , & x > L . \end{cases}$$

The coefficients r and t refer to the _reflected_ and _transmitted_ part of the _incident_ wave e^{ikx}.

(!)

That the scattering problem (†) has a _unique_ _solution_ can be shown by means of ODE theory or within the context of Sobolev space theory and coercive forms. We leave this for later. For now, observe that the scattered field u satisfies

induced forcing

$F[w_0](x)$

$$-\sigma(\tau u')' - k^2 u = \sigma(\tau w_0')' + k^2 w_0$$
$$= \left(\sigma(\tau w_0')' - w_0'' \right) \chi_{[-L, L]} =: F[w_0](x)$$

in which the right-hand side is viewed as a forcing function imposed by the incident field $w_0(x, k)$. This _forcing_ is _confined to the obstacle_ in $[-L, L]$.

Distorted Fourier transform relative to scattering states.

For $\psi \in L^2(\mathbb{R})$, define

$$\hat{\psi}_1(k) := \int \psi(x)\, \bar{w}(x;k)\, dx$$

(!) This is a bounded operator from L^2 to L^2 and is unitarily equivalent to the usual Fourier transform.

3. <u>Modified scattering states</u> (<u>modified distorted waves</u>)

For $\lambda \in \mathbb{C}$ with $\operatorname{Im}\lambda \geq 0$, define the fields $w(x;k,\lambda)$ by

$$-\sigma(\tau u')' - \lambda u = F[w_0] \qquad \left(= \sigma(\tau w_0')' - w_0''\right)$$
$$w(x;k,\lambda) = w_0(x;k) + u(x;k,\lambda) \qquad \overset{\uparrow}{-k^2 w_0(x;k)}$$
$$u(x;k,\lambda) = \begin{cases} u_+ e^{i\sqrt{\lambda}\,x} & \text{for } x > L \\ u_- e^{-i\sqrt{\lambda}\,x} & \text{for } x < -L \end{cases}$$

in which $\sqrt{\lambda}$ is in the $\overset{\text{closed}}{\text{first}}$ quadrant. If $\operatorname{Im}\lambda > 0$, $u(x;k,\lambda) \to 0$ as $|x| \to \infty$. This

(!) problem has a <u>unique solution</u>.

The importance of $w(x;k,\lambda)$ lies in the equation

$$\overset{(*)}{\longrightarrow} \qquad (-\sigma(\partial_x \tau \partial_x) - \lambda)\, w(x;k,\lambda) = (k^2 - \lambda)\, w_0(x;k).$$

Modified distorted Fourier transform:
For $\psi \in L^2(\mathbb{R})$,

$$\hat{\psi}_1(k, \lambda) := \int \psi(x) \, \overline{w}(x; k, \lambda) \, dx$$

4. **Formula for the resolvent of $-\sigma \partial \tau \partial$**

Because of (*), we have

$$(-\sigma \partial \tau \partial - \lambda) \frac{w(x; k, \lambda)}{k^2 - \lambda} = w_0(x; k)$$

Thus,

$$(-\sigma \partial \tau \partial - \lambda) \int \frac{\hat{\psi}_0(k)}{k^2 - \lambda} w(x; k, \lambda) \, dk$$

(!) needs to be justified

$$\overset{!}{=} \int \hat{\psi}_0(k) \frac{(-\sigma \partial \tau \partial - \lambda) w(x; k, \lambda)}{k^2 - \lambda} \, dk$$

$$= \int \hat{\psi}_0(k) \, w_0(x; k) \, dk = \psi(x).$$

This means that

✓

$$(R_\lambda \psi)(x) := \int \frac{\hat{\psi}_0(k)}{k^2 - \lambda} w(x; k, \lambda) \, dk \, .$$

Next, we find the Fourier transform of $R_{\bar{\lambda}} \psi$.

Compute $(R_{\bar{\lambda}}\varphi)_0^\wedge(k) = \dfrac{\hat{\varphi}_1(k,\lambda)}{k^2-\bar{\lambda}}$ for $\text{Im }\lambda < 0$

Let $\varphi \in L^2(\mathbb{R})$ be given:

$$(R_{\bar{\lambda}}\varphi, \psi) = (\varphi, R_\lambda\psi) = \int \varphi(x)\overline{(R_\lambda\psi)(x)}\,dx$$

(!)

$$= \int \varphi(x) \int \frac{\overline{\hat{\psi}_0(k)}}{k^2-\bar{\lambda}}\,\overline{w}(x;k,\lambda)\,dk\,dx$$

$$= \int \frac{\overline{\hat{\psi}_0(k)}}{k^2-\bar{\lambda}} \int \varphi(x)\,\overline{w}(x;k,\lambda)\,dx\,dk$$

$$= \int \frac{\overline{\hat{\psi}_0(k)}}{k^2-\bar{\lambda}}\,\hat{\varphi}(k,\lambda)\,dk = \left(\frac{\hat{\varphi}(k,\lambda)}{k^2-\bar{\lambda}},\,\hat{\psi}_0(k)\right)$$

By the unitarity of the Fourier transform,

✓

$$(R_{\bar{\lambda}}\varphi)_0^\wedge(k) = \frac{\hat{\varphi}_1(k,\lambda)}{k^2-\bar{\lambda}}$$

5. Obtain the <u>spectral measures</u> for $-\sigma\partial\tau\partial$. ← different σ!

First, we derive a formula for the functions $\sigma(t;\varphi)$

Let R_z be the resolvent of a self-adjoint operator, and let $a < b$ be given.

$$\lim_{\varepsilon \to 0} \int_a^b \varepsilon\|R_{t\pm i\varepsilon}\varphi\|^2\,dt = \lim_{\varepsilon \to 0}\int_a^b \varepsilon \int_{-\infty}^{\infty} \frac{1}{t-(\tau\pm i\varepsilon)}\,d(E_\tau\varphi, R_{\tau\pm i\varepsilon}\varphi)\,dt$$

$$= \lim_{\varepsilon \to 0} \int_a^b \varepsilon \int_{-\infty}^{\infty} \frac{1}{t-(\tau \pm \varepsilon)} \, d(R_{\tau \mp i\varepsilon} E_t \varphi, \varphi) \, d\tau$$

$$= \lim_{\varepsilon \to 0} \int_a^b \varepsilon \int_{-\infty}^{\infty} \frac{1}{t-(\tau \pm i\varepsilon)} \, d_t \int_{-\infty}^{\infty} \frac{1}{s-(\tau \mp i\varepsilon)} \, d_s (E_s E_t \varphi, \varphi) \, d\tau$$

$$= \lim_{\varepsilon \to 0} \int_a^b \varepsilon \int_{-\infty}^{\infty} \frac{1}{t-(\tau \pm i\varepsilon)} \, d_t \int_{-\infty}^{t} \frac{1}{s-(\tau \mp i\varepsilon)} \, d_s (E_s \varphi, \varphi) \, d\tau$$

$$= \lim_{\varepsilon \to 0} \int_a^b \varepsilon \int_{-\infty}^{\infty} \frac{1}{|t-(\tau \pm i\varepsilon)|^2} \, d(E_t \varphi, \varphi) \, d\tau$$

$$= \lim_{\varepsilon \to 0} \int_a^b \int_{-\infty}^{\infty} \frac{\varepsilon}{(t-\tau)^2 + \varepsilon^2} \, d\sigma(t; \varphi) \, d\tau$$

(!)
$$= \int_{-\infty}^{\infty} \lim_{\varepsilon \to 0} \int_a^b \frac{\varepsilon}{(t-\tau)^2 + \varepsilon^2} \, d\tau \, d\sigma(t; \varphi)$$

$$= \int_{-\infty}^{\infty} \left(\pi \chi_{(a,b)} + \frac{\pi}{2} \left(\chi_{\{a\}} + \chi_{\{b\}} \right) \right) d\sigma(t; \varphi)$$

$$= \frac{\pi}{2} \left[\left(\sigma(b+0; \varphi) + \sigma(b-0; \varphi) \right) - \left(\sigma(a+0; \varphi) - \sigma(a-0; \varphi) \right) \right]$$

$$\left[\text{if } \sigma \text{ is continuous} = \pi \left(\sigma(b; \varphi) - \sigma(a; \varphi) \right) \right]$$

av. of rt & lf. limits

(*) So $\displaystyle \lim_{\varepsilon \to 0} \int_a^b \varepsilon \| R_{\tau \pm i\varepsilon} \varphi \|^2 \, d\tau = \pi \left(\overline{\sigma}(b; \varphi) - \overline{\sigma}(a; \varphi) \right)$

Now, using the formula $(R_{\bar{\lambda}}\varphi)^{\wedge}_0(k) = \frac{\hat{\varphi}_1(k,\lambda)}{k^2-\bar{\lambda}}$,

we compute

$$\lim_{\varepsilon\to 0}\int_a^b \varepsilon\|R_{\tau-i\varepsilon}\varphi\|^2\,d\tau = \lim_{\varepsilon\to 0}\int_{-\infty}^{\infty}\int_a^b \frac{\varepsilon}{(k^2-\tau)^2-\varepsilon^2}\left|\hat{\varphi}_1(k,\tau+i\varepsilon)\right|^2\,d\tau\,dk$$

Now use the fact (needs to be proved) that $|\hat{\varphi}_1(k,\tau+i\varepsilon)|$ is continuous in τ and ε for $\varepsilon \geq 0$, and that

(!)
$$\lim_{\varepsilon\to 0}\hat{\varphi}_1(k,k^2+i\varepsilon) = \hat{\varphi}_1(k)$$

$$\longrightarrow = \pi\int_{-\infty}^{\infty}\left|\hat{\varphi}_1(k)\right|^2\chi_{\{a\leq k^2\leq b\}}\,dk = \pi\int_{a\leq k^2\leq b}\left|\hat{\varphi}_1(k)\right|^2\,dk$$

Since $k^2\geq 0$ $\forall k$, we may make $a=0$ and $b\geq 0$:

$$\longrightarrow = \pi\left(\int_{-\sqrt{b}}^{-\sqrt{a}}+\int_{\sqrt{a}}^{\sqrt{b}}\right)\left|\hat{\varphi}_1(k)\right|^2\,dk$$

$$= \pi\int_{\sqrt{a}}^{\sqrt{b}}\left(\left|\hat{\varphi}_1(k)\right|^2+\left|\hat{\varphi}_1(-k)\right|^2\right)\,dk$$

by (*) p.53
$$= \pi\int_{\bar{a}}^{\bar{b}}d\sigma(\lambda;\varphi)\left(= \pi\int_a^b \sigma'(\lambda;\varphi)\,d\lambda \quad \text{b/c } \sigma \text{ abs. cont.}\right)$$

So with $k^2=\lambda$, we have $d\sigma(\lambda;\varphi) = \left(\left|\hat{\varphi}_1(k)\right|^2+\left|\hat{\varphi}_1(-k)\right|^2\right)\,dk$.

6. Letting $a \to 0$ and $b \to \infty$, we obtain

$$\int_{-\infty}^{\infty} |\hat{e}_1(k)|^2 \, dk = \int_{\underset{\sim 0}{-\infty}}^{\infty} d\sigma(\lambda; e, e) = (e, e)$$

so that $e \mapsto \hat{e}$ preserves the norm. In fact this map is $\underline{invertible}$, as we now show.

From the result

Can take
$0 \leq a$

$$\int_{a \leq k^2 \leq b} |\hat{e}_1(k)|^2 \, dk = \int_a^b d\sigma(\lambda; e, e) = ((E_b - E_a)e, e)$$

and the polarization procedure, we obtain

$$\left((E_b - E_a)f, \psi\right) = \int_{a \leq k^2 \leq b} \hat{f}_1(k) \overline{\hat{\psi}(k)} \, dk \qquad \text{for } \psi \in C_c^{\infty}(\mathbb{R})$$

$$= \int_{a \leq k^2 \leq b} \hat{f}_1(k) \int_{-\infty}^{\infty} \overline{\psi(x)} \, w(x; k) \, dx \, dk$$

(!)
$$= \int_{-\infty}^{\infty} \overline{\psi(x)} \int_{a \leq k^2 \leq b} \hat{f}_1(k) \, w(x; k) \, dk \, dx$$

$$\Rightarrow \quad (E_b - E_a) f(x) = \int_{a \leq k^2 \leq b} \hat{f}_1(k) w(x; k) \, dk \qquad \Downarrow \Downarrow \Downarrow$$

Integral representation of f

Taking $a \to 0$ and $b \to \infty$, we obtain

$$f(x) = \int_{-\infty}^{\infty} \hat{f}_1(k) w(x; k) \, dk$$