Source: Kolmogorov & Fomin §36 (for example)

Functions of bounded variation and the Stieltjes integral

Recall that the total variation of a function $\sigma : [a,b] \to \mathbb{R}$ is defined by

$$V_a^b(\sigma) = \sup_{a \le x_0 < \dots < x_N = b} \sum_{i=1}^{N} |\sigma(x_i) - \sigma(x_{i-1})| .$$

The supremum is taken over all partitions $a \le x_0 < x_1 < \dots < x_N = b$ of $[a,b]$.

If $V_a^b(\sigma) < \infty$, then σ is said to be of bounded variation on $[a,b]$.

Recall that $V_a^b(\sigma) < \infty$ if and only if σ is the difference of two increasing functions on $[a,b]$:

$$\sigma(x) = \sigma_+(x) - \sigma_-(x) , \quad x \in [a,b] , \ \sigma_+ \uparrow , \ \sigma_- \uparrow .$$

[In this class, σ is increasing if $x < y \Rightarrow \sigma(x) \le \sigma(y) \ \forall x,y \in [a,b]$]

If $V_a^b(\sigma) < \infty$, then σ has left and right limits at each point $x \in [a,b]$:

$$\sigma(x-) := \lim_{\substack{y \to x \\ y \in [a,x)}} \sigma(y) , \quad \sigma(x+) := \lim_{\substack{y \to x \\ y \in (x,b]}} \sigma(y)$$

and the points of discontinuity of σ $(\sigma(x-) \ne \sigma(x+))$ form a countable set.

The Riemann-Stieltjes integral

For a <u>partition</u> $P: a \le x_0 < x_1 < \dots < x_N = b$ of $[a,b]$, define $\|P\| = \max_{i=1}^{N} |x_i - x_{i-1}|$. A <u>sampled partition</u> a partition together with sampling points $\hat{x}_i \in [x_{i-1}, x_i]$; $i = 1, \dots, N$: $\hat{P} = (P, \{\hat{x}_i\}_{i=1}^N)$. Define $\|\hat{P}\| = \|P\|$

The <u>Riemann-Stieltjes integral</u> of a function $f: [a,b] \to \mathbb{C}$ ($\mathbb{C}$ may be replaced with any complex vector space) with respect to a function $\sigma: [a,b] \to \mathbb{R}$ of bounded variation is defined to be

$$\int_a^b f(x)\, d\sigma(x) := \lim_{\|\hat{P}\| \to 0} \sum_{i=1}^{N} f(\hat{x}_i)\left(\sigma(x_i) - \sigma(x_{i-1})\right)$$

whenever this limit over all sampled partitions exists.

The Lebesgue-Stieltjes integral

Given a function $\sigma: [a,b] \to \mathbb{R}$ that is <u>increasing</u> define a measure μ_σ to be the <u>Lebesgue extension of the measure m defined on subintervals of $[a,b]$</u> by

$$m(\alpha,\beta) = \sigma(\beta-) - \sigma(\alpha+)$$
$$m[\alpha,\beta] = \sigma(\beta+) - \sigma(\alpha-)$$
$$m[\alpha,\beta) = \sigma(\beta-) - \sigma(\alpha-)$$
$$m(\alpha,\beta] = \sigma(\beta+) - \sigma(\alpha+)$$

Now let σ be of bounded variation (not necessarily increasing), and let σ_+ be the "total variation function" of σ:

$$\sigma_+(x) = V_a^x(\sigma), \quad x \in [a,b],$$

and put $\sigma_-(x) = \sigma_+(x) - \sigma(x)$. This gives the canonical decomposition of σ into the difference of two increasing functions,

$$\sigma(x) = \sigma_+(x) - \sigma_-(x), \qquad x \in [a,b]$$

(one can verify that σ_- is increasing).

The Lebesgue-Stieltjes integral of $f : [a,b] \to \mathbb{C}$ with respect to σ is defined as

$$\int_a^b f(x)\, d\sigma(x) := \int_{[a,b]} f\, d\mu_{\sigma_+} - \int_{[a,b]} f\, d\mu_{\sigma_-},$$

for all μ_{σ_+} and μ_{σ_-} integrable functions, which includes Borel-measurable functions.

Fact If $f : [a,b] \to \mathbb{C}$ is continuous, then both forms of the Stieltjes integral exist and coincide.

<u>The Helly selection theorem</u> Let $\sigma_n : [a,b] \to \mathbb{R}$, $n \in \mathbb{N}$, be a sequence of uniformly bounded increasing functions, say $|\sigma_n(x)| \le M$ $\forall$ $x \in [a,b]$ $\forall$ $n \in \mathbb{N}$. Then there exists a subsequence $\{\sigma_{n_j} : j \in \mathbb{N}\}$ and an increasing function $\sigma : [a,b] \to \mathbb{R}$ such that $\sigma_{n_j}(x) \to \sigma(x)$ $(j \to \infty)$ $\forall$ $x \in [a,b]$.

<u>Proof</u> Let S be a countable dense subset of $[a,b]$. Thus there is a subsequence $\{\sigma_{n_j} : j \in \mathbb{N}\}$ such that $\sigma_{n_j}(x)$ converges for each $x \in S$; denote the limit by $\sigma(x)$. (Proof of this is an exercise.) Define, $\forall$ $x \in [a,b]$,

$$\underline{\sigma}(x) := \liminf_{j \to \infty} \sigma_{n_j}(x), \quad \overline{\sigma}(x) := \limsup_{j \to \infty} \sigma_{n_j}(x).$$

Since $\forall$ $x \in S$, $\underline{\sigma}(x) = \sigma(x) = \overline{\sigma}(x)$ and since S is dense in $[a,b]$, we find that $\underline{\sigma}(x) = \overline{\sigma}(x)$ at all points at which both $\underline{\sigma}$ and $\overline{\sigma}$ are continuous. At all such points, set $\sigma(x) = \underline{\sigma}(x) = \overline{\sigma}(x) = \lim_{j \to \infty} \sigma_{n_j}(x)$.
Set $S' = \{x \in [a,b] : \underline{\sigma}$ or $\overline{\sigma}$ is discontinuous at $x\}$.
Since both $\underline{\sigma}$ and $\overline{\sigma}$ are increasing, S' is countable.
Let $\{\sigma_{n_j'} : j \in \mathbb{N}\}$ be a subsequence of $\{\sigma_{n_j} : j \in \mathbb{N}\}$ such that $\sigma_{n_j'}(x)$ converges for each $x \in S'$, and again denote the limit by $\sigma(x)$, so that

$$\sigma(x) = \lim_{j \to \infty} \sigma_{n_j'}(x) \quad \forall \ x \in [a,b];$$

σ is increasing because each function $\sigma_{n_j'}$ is increasing. ∎

Recall the following fact (related to the Riesz representation theorem for $C[a,b]$, the space of continuous fcns on $[a,b]$):

<u>Fact</u> If $f:[a,b]\to\mathbb{C}$ is continuous and $\sigma:[a,b]\to\mathbb{R}$ is of bounded variation, then

$$\left|\int_a^b f(x)\,d\sigma(x)\right| \le V_a^b(\sigma)\,\|f\|_{\sup}.$$

<u>The Helly convergence ~~theorem~~</u>

Let $\{\sigma_n:[a,b]\to\mathbb{R}\}_{n\in\mathbb{N}}$ be a sequence of functions that converges pointwise to a function σ, suppose that $V_a^b(\sigma_n)\le M$ $\forall\, n\in\mathbb{N}$, and let $f:[a,b]\to\mathbb{C}$ be continuous. Then $V_a^b(\sigma)\le M$ and

$$\int_a^b f(x)\,d\sigma_n(x) \longrightarrow \int_a^b f(x)\,d\sigma(x) \quad \text{as } n\to\infty.$$

<u>Proof</u> To show $V_a^b(\sigma)\le M$, let P be any partition of $[a,b]$. Then

$$\sum_{i=1}^N |\sigma(x_i)-\sigma(x_{i-1})| = \lim_{n\to\infty}\sum_{i=1}^N |\sigma_n(x_i)-\sigma_n(x_{i-1})| \le M,$$

and thus $V_a^b(\sigma)\le M$.

Let $\varepsilon>0$ be given, and let $g:[a,b]\to\mathbb{C}$ be a step function characterized by a partition $a=x_0<x_1<\dots<x_m=b$, and the numbers $g_k\in\mathbb{C}$, $k=1,\dots,m$, with $g(x)=g_k$ for $x_{k-1}\le x<x_k$, $k=1\dots m$. and $g(b)=g_M$ Let the $\{x_k\}$ and $\{g_k\}$ be chosen such that two properties hold:

(1) σ and σ_n $\forall\, n \in \mathbb{N}$ are continuous at each x_k, $k \neq 0, m$

(2) $\|f - g\|_{\sup} < \frac{\varepsilon}{3M}$.

[Property (1) is possible since each σ_n as well as σ has only countably many points of discontinuity ..., Property (2) is possible because f is continuous.]

Then we obtain

$$\left| \int_a^b f\, d\sigma - \int_a^b g\, d\sigma \right| \leq V_a^b(\sigma)\, \|f-g\|_{\sup} < \frac{\varepsilon}{3}$$

$$\left| \int_a^b f\, d\sigma_n - \int_a^b g\, d\sigma_n \right| < \frac{\varepsilon}{3}$$

Because σ_n and σ are continuous at x_k, $k \neq 0, m$

$$\int_a^b g\, d\sigma = \sum_{k=1}^{m} g_k\, \mu_\sigma [x_{k-1}, x_k) + g_m(\sigma(b) - \sigma(b-))$$

$$= \sum_{k=1}^{m} g_k \left(\sigma(x_k) - \sigma(x_{k-1})\right)$$

and likewise $\int_a^b g\, d\sigma_n = \sum_{k=1}^{m} g_k \left(\sigma_n(x_k) - \sigma_n(x_{k-1})\right)$ $\forall\, n \in \mathbb{N}$.

By assumption, $\sigma_n \to \sigma$ pointwise, so $\exists\, n_0$ s.th. if $n \geq n_0$ then

$\left| \int_a^b g\, d\sigma - \int_a^b g\, d\sigma_n \right| < \varepsilon/3$, and we obtain

$\left| \int_a^b f\, d\sigma - \int_a^b f\, d\sigma_n \right| < \varepsilon$ $\forall\, n \geq n_0$. ∎