1D Extreme simplicity of continuous flows in 1D . . .

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2D Simplicity of continuous flows in 2D.

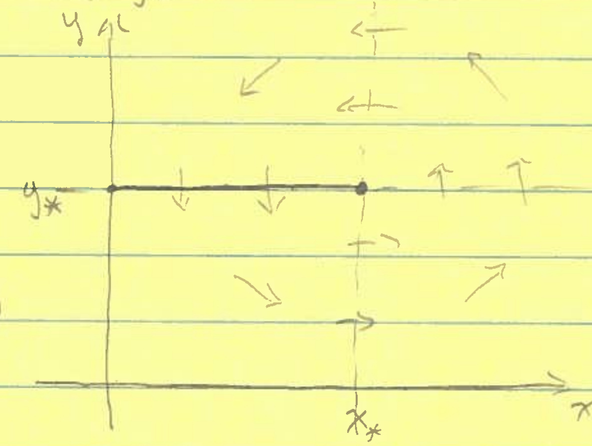
ex. Lotka Volterra model with a prey source:
$$\begin{cases} \dot{x} = kx - axy + sy \\ \dot{y} = -ly + bxy \end{cases}$$

Eq. pts: $(x,y) = (0,0)$ & $(x,y) = \left(\frac{l}{b}, \frac{k}{a - s\frac{b}{l}}\right)$

In general, consider a similar system with the coordinate axes as isoclines and with equilibrium points $(0,0)$ and $(x_*, y_*)$.

Phase plane →

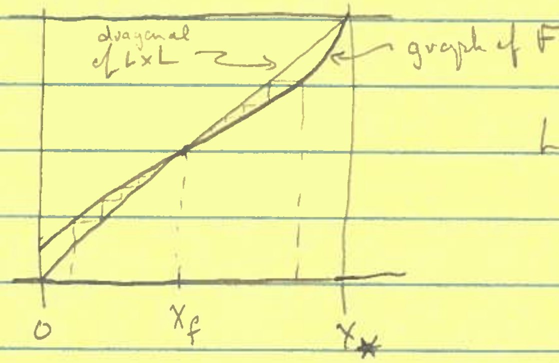
Cross-section segment
$L = ((0, x_*), y_*)$




(1) It is reasonable that, for appropriate systems, one could prove that any trajectory that begins on $L$ returns to $L$.
(2) Given this is true, let $(x_0, y_*) \in L$ be an initial condition, let $(x_1, y_*)$ be the first return of the solution to $L$, $(x_2, y_*)$ the second return, ..., $(x_n, y_*)$ the $n^{th}$ return. This generates a discrete dynamical system
$$x_{n+1} = F(x_n),$$
where $F$ is the return map. $F$ is called the Poincaré map (for this ODE and the segment $L$).

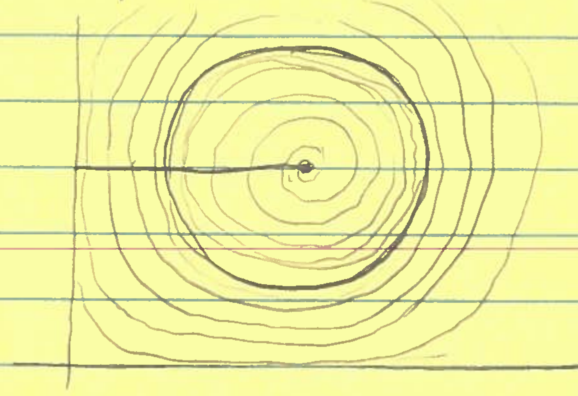
(3) One can prove that $F$ is strictly increasing and continuous, with fixed point at $x_*$

(*)




Thus, the Poincaré map is a discrete dynamical system that is easily visualized graphically by a "cobweb", or "Lamerey" diagram.

(4) Fixed points of $F$ $(F(x_f) = x_f)$ correspond to periodic orbits of the ODE.



In figure (*), $x_*$ is unstable for $F$, so $(x_*, y_*)$ is unstable for the ODE. $x_f$ is stable for $F$, so the periodic orbit through $(x_f, y_*)$ is <u>attracting</u> for the ODE.

Q: When does an ODE system in 2D admit a Poincaré map?
This question is addressed by the Poincaré-Bendixson Theorem.

(5) First we define the $\omega$-limit (omega-limit) $\omega[\gamma]$ of an orbit $\gamma$ [$\gamma(t) = (x(t), y(t))$ soln of ODE]:

$\omega[\gamma]$ is the set of all points $p=(x,y)$ such that there exists a sequence $\{t_n\}_{n=1}^{\infty}$ of increasing times tending to $\infty$ with $\gamma(t_n) \to p$ as $n \to \infty$.

(6) <u>Poincaré-Bendixson Theorem</u>
Let $\dot{\underline{x}} = G(\underline{x})$ be an ODE in $\mathbb{R}^2$ with $G$ continuously differentiable, and let $\gamma$ be an orbit thereof.
Then $\omega[\gamma]$ has one of the following properties:

- $\omega[\gamma]$ is empty
- $\omega[\gamma]$ has infinitely many fixed points
- $\omega[\gamma]$ consists of a single fixed point
- $\longrightarrow$ $\omega[\gamma]$ is a periodic orbit.
- $\omega[\gamma]$ consists of finitely many fixed points connected by homoclinic or heteroclinic orbits.

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