Math 7320 @ LSU
Spring, 2018
Problem Set 2

Textbook problems refer to section number, subsection number, and problem number in V.I. Arnold's book "Ordinary Differential Equations".

1. This problem describes Putzer's algorithm for computing the solution of a linear constant-coefficient homogeneous system $\dot{x} = Ax$ without first computing the Jordan normal form for the generator A. Let A be a linear operator in a finite-dimensional complex vector space of dimension n, and let $\{\lambda_i\}_{i=1}^{n}$ denotes its eigenvalues. For $k = 1, \ldots, n$, define the operators

$$A_k = \prod_{j=1}^{k} (A - \lambda_j E)$$

and the scalar functions $r_j(t)$ as the solution of the system

$$dr_1/dt = \lambda_1 r_1, \qquad r_1(0) = 1$$
$$dr_j/dt = \lambda_j r_j + r_{j-1}, \quad r_j(0) = 0 \quad \text{for } j \geq 2.$$

Putzer's theorem states that

$$e^{tA} = \sum_{k=0}^{n-1} r_{k+1}(t) A_k.$$

2. Text: 20.6.1

3. Consider the linear non-autonomous nonhomogeneous linear n^{th}-order initial-value ordinary differential equation

$$x^{(n)}(t) + \sum_{\ell=1}^{n} a_\ell(t) \, x^{(n-\ell)}(t) = F(t)$$

$$x^{(\ell)}(t_0) = x_\ell, \quad (0 \leq \ell < n)$$

$$(1)$$

Set $\phi(t) = x^{(n)}(t)$. Prove that, if $x(t)$ satisfies (1), then $\phi(t)$ satisfies the Volterra integral equation

$$\phi(t) + \int_{t_0}^{t} K(t,s)\phi(s) = f(t),$$

in which the kernel K is defined by

$$K(t,s) = \sum_{\ell=1}^{n} a_\ell(t) \frac{(t-s)^{\ell-1}}{(\ell-1)!}$$

and the term of inhomogeneity is

$$f(t) = F(t) - \sum_{\ell=1}^{n} a_\ell(t) \left(c_{n-1} \frac{t^{\ell-1}}{(\ell-1)!} + \cdots + c_{n-\ell} \right).$$

4. Consider the spectral Volterra-integral problem

$$\phi(t) - \lambda \int_{t_0}^t K(t,s)\phi(s)\,ds \;=\; f(t) \qquad (t_0 \le t \le t_1),$$

in the L^2 theory. Prove that, for fixed f, the solution ϕ is analytic in λ and satisfies a Volterra integral equation

$$\phi(t) \;=\; f(t) - \lambda \int_{t_0}^t H(t,s;\lambda)f(s)\,ds \qquad (t_0 \le t \le t_1).$$

Find the kernel $H(t,s;\lambda)$ in terms of the kernel $K(t,s)$.