Math 7320 @ LSU
Spring, 2018
Problem Set 3

Textbook problems refer to section number, subsection number, and problem number in V.I. Arnold's book "Ordinary Differential Equations".

1. Consider the initial-value problem on the interval $I = (t_0 - a, t_0 + a)$

$$\frac{d}{dt}x(t) = A(t)x(t) + f(t), \qquad t \in I$$

$$x(t_0) = x_0,$$

in which $A \in L^2(I)$ and $f \in L^1(I)$. Prove that this problem has a unique solution in $L^2(I)$ and that this solution is absolutely continuous.

2. Let the transfer matrix for the ordinary differential equation

$$\frac{d}{dt}x(t) = \big(A_0(t) + \sigma A_1(t)\big)x(t)$$

be denoted by $X_\sigma(t, t_0)$. This means that the solutions satisfy

$$x(t) = X_\sigma(t, t_0)\, x(t_0).$$

Prove that

$$X_\sigma(t, t_0) = X_0(t, t_0) + \sigma \int_{t_0}^{t} X_0(t, s)A_1(s)X_0(s, t_0)ds + O(\sigma^2) \qquad (\sigma \to 0).$$

Prove this rigorously, paying attention to the precise definition of the "big O" symbol $O(\sigma)$.

3. Consider the initial-value problem

$$-u'' + q(x)u = f(x),$$
$$u(0) = a, \quad u'(0) = b, \qquad x \in I = [0, \pi], \qquad (1)$$

in which $f \in L^1(I)$ and $q \in L^2(I)$. Prove that this problem has a unique absolutely continuous solution that satisfies

$$u(x) - \int_0^x (x-y)q(y)u(y)\, dy = a + bx - \int_0^x (x-y)f(y)\, dy.$$

Prove that, given $C > 0$, there exists a real number M such that

$$\|q\|_2 < C \implies \|u\|_2 < M$$

and a real number N such that

$$\|q\|_2 < C \implies \max_{x \in I}|u(x)| < N.$$

4.   Consider the initial-value problem

$$-u_\epsilon'' + \big(q(x) + \epsilon\big)u_\epsilon \; = \; 0\,,$$
$$x \in I = [0, \pi]\,,$$
$$u_\epsilon(0) = a\,, \quad u_\epsilon'(0) = b\,,$$

in which a and b are fixed complex numbers and ϵ is a complex variable.

a.   Prove that, for each $\rho > 0$, there are numbers M and N such that

$$|\epsilon| < \rho \implies \|u_\epsilon\|_2 < M$$

and

$$|\epsilon| < \rho \implies \max_{x \in I} |u_\epsilon(x)| < N.$$

b.   By deriving an initial-value problem of the form (1) for the function $u_\epsilon - u_0$, prove that the map $\epsilon \mapsto u \in L^2(I)$ is a continuous function.

c.   By deriving an initial-value problem of the form (1) for the function $(u_\epsilon - u_0)/\epsilon$, use the causal Green function for the operator $-d^2/dx^2 + q(x)$ to prove that, for each $x \in I$, $u_\epsilon(x)$ is differentiable with respect to the complex variable ϵ at $\epsilon = 0$.

5.   Prove that, for the solution $u_\epsilon(x)$ of the previous initial-value problem, the derivatives $u_\epsilon'(x)$ are analytic in ϵ at $\epsilon = 0$.
Suggestion: Using the DE, write an integral expression for $u'(x)$ that involves u. Using the results of previous problem, this will allow you to bound $u'(x)$ pointwise, even over all ϵ in any bounded set. Then use the analyticity of $u_\epsilon(x)$ (for *all* ϵ) and a theorem from complex analysis, *etc.*