Math 7320 @ LSU
Spring, 2018
Problem Set 4

Textbook problems refer to section number, subsection number, and problem number in V.I. Arnold's book "Ordinary Differential Equations".

1. Let J be a finite interval in the real line, and let B be the ball $B = \{y \in \mathbb{R}^n \mid |y - x_0| < \rho\}$, where $x_0 \in \mathbb{R}^n$ and $\rho > 0$ are fixed. Prove that the set of functions

$$X = \{x : J \to B \mid x \text{ is continuous}\}$$

endowed with the supremum norm

$$\|x_1 - x_2\|_{\text{sup}} = \sup_{t \in J} |x_1(t) - x_2(t)|$$

is a complete metric space.

2. Let S be a subset of $\mathbb{R}^n$. Prove the following. If a function $F : S \to \mathbb{R}^m$ is of Lipschitz class, then F is continuous, and, if S is also bounded, then the range of F is bounded.

3. Let X be defined as in problem (1), let t_0 be in J, and let $F : B \to \mathbb{R}^n$ be of Lipschitz class. Prove that, for each function $x \in X$, the following two conditions are equivalent.

(IVP) x is differentiable at each point of J, $\dot{x} = F(x)$, and $x(t_0) = x_0$.

(IE) $x(t) = x_0 + \int_{t_0}^{t} F(x(s))\, ds$.

4. Solve the ODE

$$\frac{dx}{dt} = x^2 - 4xt + 4t^2$$

according to the ideas in the notes "Vector fields vs. direction fields".

5. Devise an example of a direction field (line field, not vector field) in $\mathbb{R}^3$ whose integral curves can be obtained by integration after choosing appropriate coordinates (but which is not obviously exactly solvable in the canonical coordinates). Find the solutions as explicitly as possible.

6. Devise an example of a direction field (line field, not vector field) in $\mathbb{R}^3$ whose integral curves are solutions of an autonomous ODE after choosing appropriate coordinates (but not using the canonical coordinates).