Arnold
Ch 1 §3

Vector fields vs. direction fields

- vector field: $F : \mathbb{R}^n \longrightarrow \mathbb{R}^n$

  ODE IVP: $\begin{cases} \dot{x} = F(x) \\ x(t_0) = x_0 \end{cases}$

  with solution = <u>a parameterized curve</u>

- line field, or direction field: $\ell : \mathbb{R}^n \longrightarrow \mathbb{RP}^{n-1} = \{\text{set of lines in } \mathbb{R}^n\}$

  A solution, or <u>integral curve</u>, of ℓ is a <u>curve</u> C (not a parameterized curve) such that

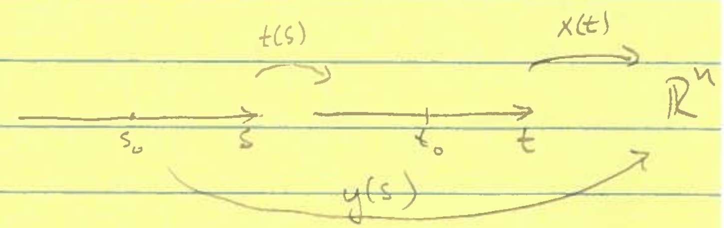
  $$TC|_x = \ell(x) \quad \text{for each } x \in C$$

  Here $TC|_x$ is the tangent space of C at $x \in C$.

  Let $x : J \to \mathbb{R}^n$ be a parameterization of C.
  Then $\dot{x}(t) \in \ell(x(t))$

  So if $F : \mathbb{R}^n \to \mathbb{R}^n$ is a vector field with $F(x) \in \ell(x)$, then the <u>image curve</u> of a solution to $\dot{x} = F(x)$ is an integral curve for ℓ.

Reparameterization:



$$\begin{cases} \dot{x}(t) = F(x(t)) \\ x(t_0) = x_0 \end{cases}$$

$$y(s) = x(t(s)) \qquad t(s_0) = t_0$$

$$\dot{y}(s) = \frac{dy}{ds}(s) = \frac{dx}{dt}(t(s))\frac{dt}{ds}(s) = F(x(t(s)))\frac{dt}{ds}(s) = F(y(s))\underbrace{\frac{dt}{ds}(s)}_{\text{scalar}} \in \ell(x)$$

Correspondence between direction fields and vector fields

(†) • Given a time-dependent vector field $F(t,x) \in \mathbb{R}^n$ (i.e. $F: \mathbb{R}^1 \times \mathbb{R}^n \to \mathbb{R}^n$) create a direction field $\ell(t,x)$ in $\mathbb{R}^{1+n}$ (i.e., $\ell: \mathbb{R}^{1+n} \to \mathbb{RP}^n$) by

$$\ell(t,x) = \text{span}(1, F(t,x))$$

Let $x(t)$ be a solution to $\dot{x} = F(t,x)$.
Then $t \longmapsto (t, x(t)) \in \mathbb{R}^{1+n}$ has tangent vector $(1, F(t,x(t)))$
So the image C of x has $TC|_{(t,x)} \in \ell(x)$.

(††) • Given a direction field $\ell: \mathbb{R}^n \to \mathbb{RP}^{n-1}$,
let $t: \mathbb{R}^n \to \mathbb{R}$ be a linear functional such that Null(t) is transverse to $\ell(x)$ for all x ($\text{Null}(t) + \ell(y) = \mathbb{R}^n \quad \forall\, y \in \mathbb{R}^n$)
Let $\{t, x^1, \ldots, x^{n-1}\}$ be a set of indep. linear fcn'ls on $\mathbb{R}^n$ (ie a lin coord system). Identify $y \in \mathbb{R}^n$ w/ its coordinates: $y \doteq (t, x^1, \ldots, x^{n-1})$. The transversality condition means that $\ell(y)$ contains a point of the form $(1, a^1, \ldots, a^{n-1})$, say $(1, F(y)) \in \ell(y)$. $F(y) = F(t,x)$ $[x = (x^1 \ldots x^{n-1})]$ is a vector field $F: \underbrace{\mathbb{R} \times \mathbb{R}^{n-1}}_{\cong \mathbb{R}^n} \to \mathbb{R}^{n-1}$. A solution of

$$\frac{dx}{dt} = F(t,x)$$

provides a parametrized curve $t \mapsto (t, x(t)) = (t, x^1(t), \ldots, x^{n-1}(t))$ in $\mathbb{R}^1 \times \mathbb{R}^{n-1}$ with tangent vector $(1, F(t,x)) \in \ell(t,x)$.
Thus the (image of) this curve is an integral curve for the dir. field ℓ.

- Direction fields with translational symmetries

(1) Let $\ell : \mathbb{R}^n \to \mathbb{RP}^{n-1}$ be <u>invariant w/resp. to a $(n-1)$-dim'l translational symmetry group</u> $X \subset \mathbb{R}^n$, that is,
$\ell(y+x) = \ell(y)$ for all $y \in \mathbb{R}^n$ and $x \in X$.
Choose a nonzero linear functional $t : \mathbb{R}^n \to \mathbb{R}$ such that $\mathrm{Null}(t) = X$. A level set of t has the form
$\{ y \in \mathbb{R}^n : t(y) = t_0 \} = x_0 + X$ (with $t(x_0) = t_0$),
and thus ℓ is invariant on level sets of t.
Suppose that ℓ is transverse to X, and apply the construction (††). Under the identification $y \doteq (t, x^1 \ldots x^{n-1})$, the invariance under translati by X is equivalent to $F(t,x) = F(t, x^1 \ldots x^{n-1})$ being independent of $x^1 \ldots x^{n-1}$, which serve as coordinates for X. Thus integral curves for ℓ are traversed by solutions to

$$\dot{x}(t) = F(t,x) = F(t).$$

The solution is now obtained by integration:

$$x(t) = x_0 + \int_{t_0}^{t} F(s)\,ds .$$

The points on the image curve C_0 are of the form

$$y \doteq (t, x(t)) = (t, x^1(t), \ldots, x^{n-1}(t)) .$$

(2) Let $\ell : \mathbb{R}^n \to \mathbb{RP}^{n-1}$ be invariant with respect to a one-dimensional translational symmetry group spanned by a vector e_0, that is, $\ell(y + te_0) = \ell(y)$ for all $y \in \mathbb{R}^n$ and all $t \in \mathbb{R}$.
Let $\{e_0, \ldots, e_{n-1}\}$ be a basis for $\mathbb{R}^n$ and $\{t, x^1, \ldots, x^{n-1}\}$ a dual basis (a coordinate system). Again make the identification $y \doteq (t, x^1, \ldots, x^{n-1})$, so that $e_0 \doteq (1, 0, \ldots, 0)$ and thus $\ell(y) = \ell(t, x)$ is independent of t.
Apply the construction from (††) again. Now, $F(t,x) = F(x)$, and integral curves of ℓ are obtained from solutions of the autonomous ODE

$$\dot{x} = F(x).$$

As we have seen before, if $x(t)$ is a solution, then so is $x(t - t_0)$ for fixed t_0.

- In the two-dimensional case of a direction field $\ell : \mathbb{R}^2 \to \mathbb{RP}^1$, cases (1) and (2) above coincide, and the constructions of F are complementary. In case (1),

$$\frac{dx}{dt} = F_1(t), \qquad x(t) = x_0 + \int_{t_0}^{t} F_1(s)\,ds.$$

In case (2),

$$\frac{dx}{dt} = F_2(x), \qquad t(x) = t_0 + \int_{x_0}^{x} \frac{1}{F_2(\xi)}\,d\xi,$$

and $x(t)$ is obtained by inverting $t(x)$.

Examples

(1) The ODE $\frac{dx}{dt} = t - x^2$ $\quad (t \in \mathbb{R},\ x \in \mathbb{R})$

specifies a slope, or direction, of a curve at each $(t,x) \in \mathbb{R}^2$.
As a line (direction) field in $\mathbb{R}^2$, it is

$$\ell(t,x) = \text{span}\{\langle 1, t-x^2 \rangle\}.$$

As a time-dependent vector field in $\mathbb{R}$, it is

$$F(t,x) = t - x^2$$

One can show that ℓ (or F) is not invariant with respect to any one-dimensional translational symmetry group.

(2) The vector field $F(t,x) = x^2 - 4xt + 4t^2 \in \mathbb{R}^1$ as a function of $(t,x) \in \mathbb{R}^2$ is invariant under translation by the vectors $a\langle 1, 2 \rangle$, $a \in \mathbb{R}$.