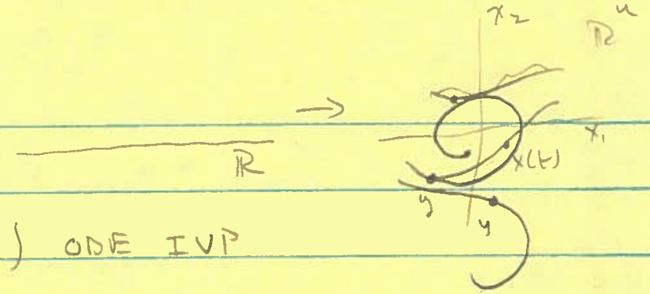
Precursor to Linear Systems (& Nonlinear)



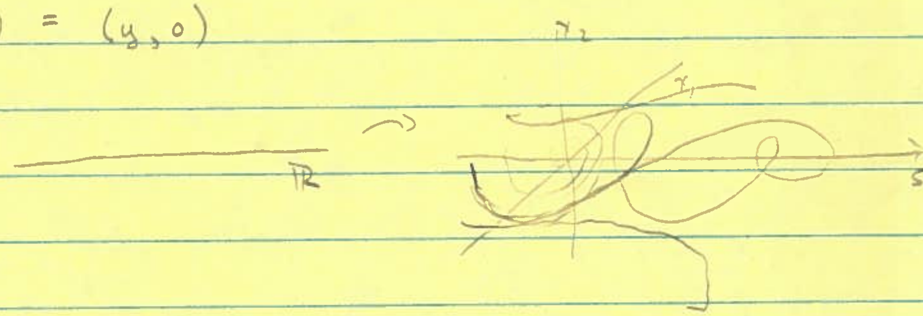
$x : \mathbb{R} \to \mathbb{R}^n$

(*) $\begin{cases} \dot{x}(t) = a(x(t), t) & \text{(nonautonomous) ODE IVP} \\ x(0) = y \end{cases}$

$\begin{cases} \dot{x}(t) = a(x(t)) & \text{autonomous ODE IVP} \\ x(0) = y \end{cases}$

In (*), introduce dep. var. s : $(x, s) \in \mathbb{R}^{n+1}$

$(*) \iff \begin{cases} \frac{d}{dt}(x(t), s(t)) = (a(x(t), t), 1) & \text{autonomous} \\ (x(0), s(0)) = (y, 0) \end{cases}$



Simplest: $\dot{x} = Ax$

Examples of (famous) ODEs and phenomena.

- Stationary 1D (nonlinear) Schrödinger equation
$$-\frac{d^2}{dx^2}\psi + \underbrace{V(x)\psi}_{\text{Lin.}} + \underbrace{|\psi|^2\psi}_{\text{NLin.}} = E\psi$$

- $\ddot{x} + \delta\dot{x} + \alpha x + \beta x^3 = \gamma \cos \omega t$ Duffing oscillator
$\rightsquigarrow$ chaos

- $\ddot{x} + \mu(x^2 - 1)\dot{x} + x = 0$ Van der Pol oscillator
$\rightsquigarrow$ periodic orbits

- $\begin{cases} \dot{x} = \sigma(y - x) \\ \dot{y} = -xz + \rho x - y \\ \dot{z} = xy - \beta z \end{cases}$ chaos and strange attractors

... etc.

②

Arnol'd §13 discusses the correspondence

$$(\dagger)\quad \{\dot{x} = Ax - \underline{\text{real}} \text{ linear auton. sys. in } \mathbb{R}^n\} \Leftrightarrow \{\text{1-param } \underline{\text{groups}} \text{ of lin. transf. of } \mathbb{R}^n\}$$

We'll look at a more general version of this for autonomous systems $\mathbb{R} \xrightarrow{x} \mathbb{R}^n \xrightarrow{a} \mathbb{R}^n$

$$\begin{cases} \dot{x} = a(x) \\ x(0) = y \end{cases}$$

Simultaneous flows $\leadsto$ family of transf. of $\mathbb{R}^n$ param. by t

$$\begin{cases} \mathbb{R} \times \mathbb{R}^n \longrightarrow \mathbb{R}^n \ :: \ (t, y) \longmapsto x(t) =: X_t y \quad [\text{or } T_t y] \\ \qquad \text{with } X_0 y = E y = y \\ \text{diff. @ } t=0 : \ \frac{d}{dt} X_t y \Big|_{t=0} = \lim_{h\to 0} \frac{1}{h}(X_h - X_0) y = \lim \frac{1}{h}(X_h y - y) =: a(y) \end{cases}$$

Property G $\quad X_{t+s} y = X_t X_s y$

Prop. A $\quad \frac{d}{dt} X_t y = a(X_t y)$

$G \Rightarrow A$: $\lim_{h\to 0} \frac{1}{h}(X_{t+h} - X_t) y = \lim_{h\to 0} \frac{1}{h}(X_h - X_0) X_t y = a(X_t y)$

$A \Rightarrow G$: Assuming uniqueness in the case that

$$\begin{cases} \frac{d}{dt} X^1_t y = a(X^1_t y) \\ X^1_0 y = y \end{cases} \ \& \ \begin{cases} \frac{d}{dt} X^2_t y = a(X^2_t y) \\ X^2_0 y = y \end{cases} \Rightarrow X^1_t y = X^2_t y$$

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(Pf) Define $\begin{cases} X^1_t y = X_{s+t}\, y \\ X^2_t y = X_t X_s\, y \end{cases}$ $\Rightarrow$ show both satisfy the same ODE IVP.

<u>Issues</u> ⊙ nonuniqueness — e.g. $\dot{x}(t) = \sqrt{|x(t)|}$

⊙ existence for only finite time (finite-time blowup)

<u>ex</u> $\dot{x}(t) = a(x(t))$, $x(0) = y$ $(n=1)$

$$t = \int_y^{x(t)} \frac{dz}{a(z)}$$

Sps $\displaystyle\int_y^{\infty} \frac{dz}{a(z)} < \infty \Rightarrow$ finite-time blowup

s.as $a(x) = x^{1+\varepsilon}$ for $x > 0$

<u>Case $a(x) = Ax$</u>, A a linear transformation — $\begin{cases} \dot{x} = Ax \\ x(0) = y \end{cases}$

<u>uniqueness</u> Sps $\dot{x}_1 = Ax_1$, $\dot{x}_2 = Ax_2$, $x_1(0) = x_2(0)$. $x := x_1 - x_2$

$\Rightarrow \dot{x} = Ax$, $x(0) = 0$, $\Rightarrow x(t) = A\int_0^t x(s)\,ds \Rightarrow |x(t)| \le |A|\,|t| \max\limits_{s(t-s)\ge 0} |x(s)|$

$\Rightarrow \max\limits_{s(t-s)\ge 0} |x(s)| \le |A|\,|t| \max\limits_{s(t-s)\ge 0} \Rightarrow x(t) = 0$ for $|t| \le |A|^{-1}$

<u>existence $\forall t$</u> The matrix ODE $\dot{X} = AX$, $X(0) = E$ has solution $X(t) = \exp(tA)$, so $\dot{x} = Ax$, $x(0) = y$ has soln $X(t)y$.

To wit: $\exp(tA) = \sum_{n=0}^{\infty} \frac{1}{n!} t^n A^n$

$\Rightarrow \forall\, T$ one has unif. absolute conv. for $|t| \le T$

& term-by-term diff. $\Rightarrow \frac{d}{dt} e^{tA} = Ae^{tA}$.

Back to (†): The correspondence is now evident.

§15 properties of the exponential.

• The group property (Prop. G) has already been proved b/c Prop. A is evident and uniqueness has been shown for $\dot{x} = Ax$.
It can also be proved from the defn. of e^{tA} as a series.

• $e^A = \lim_{m\to\infty}\left(E + \frac{A}{m}\right)^m = \lim_{m\to\infty}\left(E - \frac{A}{m}\right)^{-m}$

So $e^{tA} = \lim_{m\to\infty}\left(E + \frac{tA}{m}\right)^m$ [see p. 165 for pf.]

$\hookrightarrow$ Notice $\left(E + \frac{tA}{m}\right)^m$ is the m-fold application of the linearization $E + sA$ of e^{sA} evaluated at $s = \frac{t}{m}$.

ex $\underset{= e^{tA}}{X_t} = \begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix}$, $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

$$\underset{\text{(unstable)}}{E + \tfrac{t}{m}A} = \begin{bmatrix} 1 & -\frac{t}{m} \\ \frac{t}{m} & 1 \end{bmatrix} = \sqrt{1+\left(\tfrac{t}{m}\right)^2}\begin{bmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{bmatrix}$$

$\varphi = \arctan\frac{t}{m}$

$$\underset{\text{(stable)}}{\left(E - \tfrac{t}{m}A\right)^{-1}} = \frac{1}{1+\left(\frac{t}{m}\right)^2}\begin{bmatrix} 1 & \frac{t}{m} \\ -\frac{t}{m} & 1 \end{bmatrix} = \frac{1}{\sqrt{1+\left(\frac{t}{m}\right)^2}}\begin{bmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{bmatrix}$$

– Discussion of other Padé approx. of e^{tA}.

• $\det e^A = e^{\operatorname{tr} A}$

$\dot{x} = Ax$

- Since $\det X_t = \det e^{tA} = e^{t\,\mathrm{tr}A}$, the flow(s) of a hom. linear autonomous system scale volumes exponentially. $\mathrm{tr}A = 0 \iff$ volume is preserved by the flow.

- $\det e^{A} = \prod_{i=1}^{n} e^{\lambda_i}$, $\mathrm{tr}A = \sum_{i=1}^{n} \lambda_i$

Notice that, if A is real, then $\bar{\lambda}$ is an e val whenever λ is.

Real A in $\dot{x} = Ax$

Char. polyn. of A: $p(\lambda) = \prod_{j=1}^{m} (\lambda - \lambda_j)^{r_j} \prod_{j=1}^{\ell} [(\lambda - \mu_j)(\lambda - \bar{\mu}_j)]^{s_j}$ $\lambda_j \in \mathbb{R}$, $\mu_j \in \mathbb{C} \setminus \mathbb{R}$, $\mu_j = a_j + ib_j$

Blocks of A are similar to $\begin{bmatrix} \lambda_j & 1 & & \\ & \ddots & \ddots & \\ & & & \lambda_j \end{bmatrix}$ & $\begin{bmatrix} a & -b & I & & \\ b & a & & & \\ & & a & -b & \\ & & b & a & \\ & & & & \ddots \end{bmatrix}$ e.g.

b/c $\begin{bmatrix} a & -b \\ b & a \end{bmatrix} = T$ satisfy $(T-\mu_j)(T-\bar{\mu}_j) = 0$
ie $T^2 - 2aT + a^2 + b^2$

$A \sim \Lambda$ ie $\exists\, M$ s.th. $AM = M\Lambda$

$e^{At} = M e^{\Lambda t} M^{-1}$

$A = \sum_{j=1}^{N} \Lambda_j P_j$, P_j are complementary projections: $E = \sum_{j=1}^{N} P_j$

Λ_j are canonical-form operators

So we must find $e^{i\Lambda}$ for canonical blocks Λ.

$\Lambda =$

- $$\begin{bmatrix} a & -b \\ b & a \end{bmatrix} = a \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{E} + b \underbrace{\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}}_{J} = aE + bJ$$ ← commute

$$e^{\Lambda} = e^{aE} e^{bJ} = e^{a} \begin{bmatrix} \cos b & -\sin b \\ \sin b & \cos b \end{bmatrix} = e^{a} R(b)$$

- $\Lambda = \underbrace{\lambda E_m + N_m}_{\text{commute.}}$ $\qquad N_m = (\delta_{i+1,j})_{i,j=1}^{m}$

$$e^{t\Lambda} = e^{t\lambda} e^{tN_m} = e^{t\lambda} \underbrace{\begin{bmatrix} 1 & t & \frac{t^2}{2} & \frac{t^3}{6} \\ 0 & 1 & t & t^2/2 \\ 0 & 0 & 1 & t \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{T_m}$$

- $\Lambda = E \otimes C + N \otimes E$ in $\mathbb{R}^m \otimes \mathbb{R}^2$

$$C = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$$

$$e^{t\Lambda} = \exp(tE \otimes C)\exp(tN \otimes E) = (E \otimes e^{tC})(e^{tN} \otimes E)$$

$$= e^{tN} \otimes e^{tC} = T \otimes e^{ta} R(tb)$$

$$= e^{ta} \begin{bmatrix} R(tb) & tR(tb) & \frac{t^2}{2} R(tb) & \frac{t^3}{6} R(tb) \\ 0 & R(tb) & & \\ 0 & 0 & R(tb) & \\ 0 & 0 & 0 & R(tb) \end{bmatrix}$$