Special topic :

The ODE system for the Maxwell equations in layered media.

$E(x,y,z;t)$ = electric field $= \langle E_1, E_2, E_3 \rangle \in \mathbb{R}^3$
$H(x,y,z;t)$ = magnetic field $= \langle H_1, H_2, H_3 \rangle \in \mathbb{R}^3$

Maxwell equations :

$$(\dagger) \qquad \begin{bmatrix} 0 & \nabla\times \\ -\nabla\times & 0 \end{bmatrix} \begin{bmatrix} E \\ H \end{bmatrix} = \frac{1}{c}\frac{\partial}{\partial t} \begin{bmatrix} \varepsilon E \\ \mu H \end{bmatrix}$$

Notation: $\nabla = \langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \rangle$ grad operator in space $\mathbb{R}^3$
c = speed of waves in free space
$\varepsilon(x,y,z)$ = space-dependent dielectric tensor
$\mu(x,y,z)$ = ... magnetic tensor — 3×3 matrices

A layered, or stratified, medium is one in which ε and μ depend only on z.
Because of the invariance of the Maxwell system in x, y, and t, one can seek solutions of the form

$$(*) \qquad \begin{bmatrix} E \\ H \end{bmatrix} \longmapsto \begin{bmatrix} E(z) \\ H(z) \end{bmatrix} e^{i(k_1 x + k_2 y)} e^{-i\omega t}.$$

Here, ω is the (circular) time frequency and $\langle k_1, k_2 \rangle$ is the wavevector parallel to the layers.

Using (*) in (†) amounts to making the replacements

$$\frac{\partial}{\partial x} \longmapsto ik_1 , \quad \frac{\partial}{\partial y} \longmapsto ik_2 , \quad \frac{\partial}{\partial t} \longmapsto -i\omega .$$

Thus (†) is reduced to six scalar equations that involve at most $\frac{\partial}{\partial z}$. One can, in fact, reduce the system to a 4×4 one for the components of the fields parallel to the layers, or

$$\psi(z) = \begin{bmatrix} E_1(z) \\ E_2(z) \\ H_1(z) \\ H_2(z) \end{bmatrix} .$$

The result is the <u>ODE system</u>

$$\text{(††)} \qquad \frac{d\psi}{dz} = iJA\psi ,$$

where $J = \left[\begin{array}{cc|cc} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ \hline 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{array}\right]$ (notice $J^2 = E$)

and $A = k \left[\begin{array}{cc|cc} \varepsilon_{11} & \varepsilon_{12} & 0 & 0 \\ \varepsilon_{21} & \varepsilon_{22} & 0 & 0 \\ \hline 0 & 0 & \mu_{11} & \mu_{12} \\ 0 & 0 & \mu_{21} & \mu_{22} \end{array}\right] - k \begin{bmatrix} \varepsilon_{13} & K_2 \\ \varepsilon_{23} & -K_1 \\ -K_2 & \mu_{13} \\ K_1 & \mu_{23} \end{bmatrix} \begin{bmatrix} \varepsilon_{33}^{-1} & 0 \\ 0 & \mu_{33}^{-1} \end{bmatrix} \begin{bmatrix} \varepsilon_{31} & \varepsilon_{32} & -K_2 & K_1 \\ K_2 & -K_1 & \mu_{31} & \mu_{32} \end{bmatrix}$

where $k = \omega/c$, $K_1 = k_1/k$, $K_2 = k_2/k$.

In EM, there is a vector

$$S = \frac{c}{8\pi} E \times \overline{H} = \langle S_1, S_2, S_3 \rangle$$

called the harmonic Poynting vector
Its real part is the time-averaged energy-flux vector field associated to an EM field.

In layered media, one cares about the component of $S(z) = \frac{c}{8\pi} E(z) \times \overline{H(z)}$ perpendicular to the layers, that is, the time-averaged flux of EM energy across the layers. It turns out that this flux is given by

$$\mathrm{Re}\, S_3(z) = [\psi(z), \psi(z)],$$

in which the sesquilinear form $[\psi_1, \psi_2]$ is defined for pairs $\psi_1, \psi_2 \in \mathbb{C}^4$ by

$$[\psi_1, \psi_2] := \frac{c}{16\pi} (J\psi_1, \psi_2), \qquad \text{(flux form)}$$

in which $(\cdot, \cdot)$ is the standard complex inner product in $\mathbb{C}^4$. $[\cdot, \cdot]$ is called the flux form.

Fact $[\cdot, \cdot]$ is an indefinite form with a two-dimensional positive space and a 2-dimensional negative space.

Given that ε and μ are Hermitian tensors, JA is flux-self-adjoint, that is,

$$[JA\psi_1, \psi_2] = [\psi_1, JA\psi_2]$$

So iJA is anti-flux-self-adjoint:

$$[iJA\psi_1, \psi_2] = [\psi_1, -iJA\psi_2] .$$

We will prove later that the non-autonomous linear ODE system (H) has a unique solution given an initial condition $\psi(z_0) = \psi_0$. The flow is denoted by

$$\psi(z) = T(z_0, z)\psi_0 ,$$

where $T(z_0,z)$ is called the <u>transfer matrix</u>. It satisfies

$$T(z_1, z_2)T(z_0, z_1) = T(z_0, z_2) .$$

The flux-self-adjointness of the propagator matrix JA causes $T(z_0,z)$ to be flux-unitary, that is,

$$[T(z_0,z)\psi_1, T(z_0,z)\psi_2] = [\psi_1, \psi_2] \quad \forall \psi_1,\psi_2 \in \mathbb{C}^4.$$

<u>Exercise</u> $\nearrow$ <u>Prove this</u>.

The flux-unitarity of the flow implies a conservation of energy, that is

$$\frac{d}{dz}\left[\psi_1(z), \psi_2(z)\right] = 0,$$

where $\psi_1(z)$ and $\psi_2(z)$ are two solutions of the Maxwell ODE system.

Modes. There is a theory of operators that are self-adjoint with respect to an indefinite inner product. It generalizes the theory of self-adjoint operators in a standard (fin-dim) inner-product space. See the following reference:

I. Gohberg, P. Lancaster, L. Rodman. "Indefinite Linear Algebra and Applications", Birkhäuser Verlag 2005.

One can show that the eigenvalues of JA are real or come in complex-conjugate pairs. In the case that JA has two real eigenvalues λ_1 and λ_2 $(\lambda_1 \neq \lambda_2)$ and two complex-conjugate eigenvalues $\alpha \pm i\beta$ $(\beta > 0)$ the theory says that an eigenbasis $\{v_j\}_{j=1}^{4}$ can be chosen such that JA and $[\cdot,\cdot]$ have the form

Fix z, ω, k_1, k_2

$$JA \sim \begin{bmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & \alpha+i\beta & 0 \\ 0 & 0 & 0 & \alpha-i\beta \end{bmatrix}, \quad \left([v_i, v_j]\right)_{i,j=1}^{4} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Let us assume that $JA(z; k_1, k_2, \omega)$ is independent of z in a given interval $[z_1, z_2]$. Then the general solution of the system $\frac{d}{dz}\psi = iJA\,\psi$ in this interval is

$$\psi(z) = c_1 v_1 e^{i\lambda_1 z} + c_2 v_2 e^{i\lambda_2 z} + c_3 v_3 e^{i\alpha z} e^{-\beta z} + c_4 v_4 e^{i\alpha z} e^{\beta z}$$

These modes are classified as "rightward" or "leftward" for physical reasons:

$$\left.\begin{matrix} v_1 e^{i\lambda_1 z} \\ v_3 e^{i\alpha z} e^{-\beta z} \end{matrix}\right\} \text{ rightward in } [z_1, z_2]$$

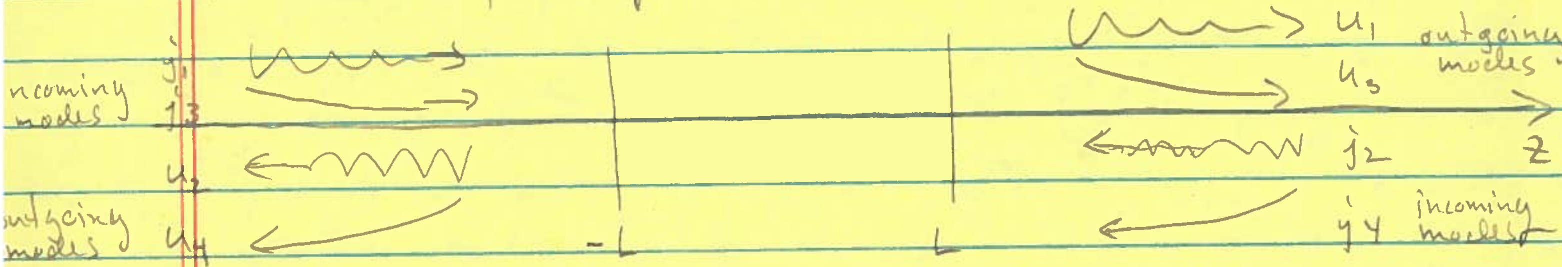
$$\left.\begin{matrix} v_2 e^{i\lambda_2 z} \\ v_4 e^{i\alpha z} e^{\beta z} \end{matrix}\right\} \text{ leftward in } [z_1, z_2].$$

Notice that $[v_1 e^{i\lambda_1 z}, v_1 e^{i\lambda_1 z}] = [v_1, v_1] = 1 > 0$, which is interpreted as the mode v_1 carrying energy to the right. $[v_2 e^{i\lambda_1 z}, v_2 e^{i\lambda_1 z}] = [v_2, v_2] = -1 < 0$ means that the mode v_2 carries energy to the left.

Since $\beta > 0$ (by assumption), $v_3 e^{i\alpha z} e^{-\beta z}$ decays as one goes rightward in z, and $v_4 e^{i\alpha z} e^{\beta z}$ decays as one goes leftward in z.

A "scattering" problem.

Let $\varepsilon(z)=\varepsilon_0$, $\mu(z)=\mu_0$ be fixed values for $|z|<L$ and ω, k_1, k_2 be chosen such that the scenario described above holds. Let $\varepsilon(z)$ and $\mu(z)$ take on other values in the "defect layer" $z\in[-L, L]$:



Problem: Given complex numbers j_1, j_2, j_3, j_4 (complex amplitudes of "incoming modes"), find complex numbers u_1, u_2, u_3, u_4 (cx amp. of "outgoing modes") such that

$$\psi(z) = j_1 v_1 e^{i\lambda_1 z} + j_3 v_3 e^{i\alpha z} e^{-\beta z} + u_2 v_2 e^{i\lambda_2 z} + u_4 v_4 e^{i\alpha z} e^{\beta z} \quad \text{for } z<-L$$

$$\psi(z) = j_2 v_2 e^{i\lambda_2 z} + j_4 v_4 e^{i\alpha z} e^{\beta z} + u_1 v_1 e^{i\lambda_1 z} + u_3 v_3 e^{i\alpha z} e^{-\beta z} \quad \text{for } z>L$$

When there exists a nonzero solution for j_1, j_2, j_3, j_4 all $=0$, that solution is known as a <u>guided mode</u> of the defect layer since one can prove that u_1 and u_2 must vanish and the field therefore decays exponentially as $|z|\to\infty$.

See two references by Shipman & Welters and other ref.s therein.