0

D.O. $Lu := -u'' + q(x)u$ on $[0,\pi]$

w/ "self-adjoint" BC:

$$(BC)\ \begin{cases} \alpha u(0) - \beta u'(0) = 0 \\ \gamma u(\pi) + \delta u(\pi) = 0 \end{cases}$$

E-val prob: Find λ s.th. $\exists$ soln u to $\begin{cases} Lu = \lambda u \\ (BC) \end{cases}$

- $\int_0^\pi (L\varphi)\overline{\psi} - \int_0^\pi \varphi(\overline{L\psi}) = W(\overline{\psi}, \varphi)\Big|_0^\pi$

- $-u'' + q(x)u = f(x)$, $u(0) = 0$, $u'(0) = u_0'$

$$\Rightarrow \quad u(x) - \int_0^x (x-y)q(y)u(y)\,dy = u_0' x - \int_0^x (x-y)f(y)\,dy$$

ie $\quad u(x) - \int_0^x K(x,y)u(y)\,dy = u_0' x + F(x)$

ie $\quad [(\mathbb{E} - \mathbb{K})u](x) = u_0' x + F(x)$

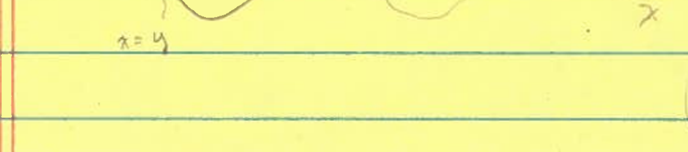
Soln: $u(x) = (\mathbb{E} + \mathbb{H})(u_0' x + F)(x)$, $\quad \mathbb{H} = \sum_{n=1}^{\infty} \mathbb{K}^n$

~ Show $\|\mathbb{E} + \mathbb{H}\|$ is bdd as fn of $\|q\|_2 \le C$.

- Green fn for L: $G(x,y)$: $\begin{cases} -\frac{\partial^2}{\partial x^2}G(x,y) + q(x)G(x,y) = \delta(x-y) \\ G(x,y) = 0 \ \text{for}\ y \le x \end{cases}$

For $q = -\lambda$: $G(x,y) = \dfrac{\sin\sqrt{\lambda}(x-y)}{\sqrt{\lambda}}$ $\quad x \ge y$

ie $\begin{cases} -\frac{\partial^2}{\partial x^2}G(x,y) + q(x)G(x,y) = 0, & x \neq y \\ G(y,y) = 0 \\ -\frac{\partial}{\partial x}G(y,y) = 1 \end{cases}$



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- $-u'' + q(x)u = f(x)$ on $\mathbb{R}$

$$\Longrightarrow \quad u(x) = \int_{\mathbb{R}} G(x,y) f(y)\,dy = \int_{-\infty}^{x} G(x,y) f(y)\,dy$$

- Perturb q by const. ε : $\begin{cases} -u_\varepsilon'' + (q(x)+\varepsilon)u_\varepsilon = 0 \\ u_\varepsilon(0) \ \& \ u_\varepsilon'(0) \text{ fixed (indep. of } \varepsilon) \end{cases}$

Show that $u_\varepsilon(x)$ is <u>analytic</u> in ε (can do at $\varepsilon = 0$).

(1) $-(u_\varepsilon - u_0)'' + q(x)(u_\varepsilon - u_0) = -\varepsilon u_\varepsilon$

$$\Longrightarrow \quad (u_\varepsilon - u_0)(x) = -\varepsilon \int_0^x G(x,y)\, u_\varepsilon(y)\,dy$$

Show continuity by uniform bound on $\|u_\varepsilon\|_2$ for ε bdd

(2) $-\left(\frac{u_\varepsilon - u_0}{\varepsilon}\right)'' + q(x)\left(\frac{u_\varepsilon - u_0}{\varepsilon}\right) = -u_0 - (u_\varepsilon - u_0)$

$$\left(\frac{u_\varepsilon - u_0}{\varepsilon}\right)(x) = -\int_0^x G(x,y)\,u_0(y)\,dy - \underbrace{\int_0^x G(x,y)(u_\varepsilon - u_0)(y)\,dy}_{\text{dep. on } \varepsilon}$$

show $\to 0$ as $\varepsilon \to$

- $-u_1'' + q(x)u_1 = 0$
  $-u_2'' + q(x)u_2 = 0$ $\Longrightarrow$ $W(u_1,u_2)(x) = \det\begin{vmatrix} u_1 & u_2 \\ u_1' & u_2' \end{vmatrix} = \text{const}$

<u>Wronskian</u> is independent of x.

$= 0$ iff u_1 & u_2 are linearly dependent.

E-val problem $\quad -u'' + q(x)u = \lambda u$

$$B_0(u) := u'(0) - h\,u(0) = 0$$
$$B_1(u) := u'(\pi) + H\,u(\pi) = 0$$

Special solns:
$$c(x,\lambda) : \quad c(0,\lambda) = 1, \quad c'(0,\lambda) = 0$$
$$s(x,\lambda) : \quad s(0,\lambda) = 0, \quad s'(0,\lambda) = 1$$
$$\varphi(x,\lambda) : \quad B_0(u) = 0, \quad u(0,\lambda) = 1 \qquad \begin{pmatrix}\varphi(0)\\ \varphi'(0)\end{pmatrix} = \begin{pmatrix}1\\ h\end{pmatrix}$$
$$\psi(x,\lambda) : \quad B_1(u) = 0, \quad u(\pi,\lambda) = 1 \qquad \begin{pmatrix}\psi(\pi)\\ \psi'(\pi)\end{pmatrix} = \begin{pmatrix}1\\ -H\end{pmatrix}$$

$$\Delta(\lambda) := W(\psi,\varphi) = \begin{vmatrix}\psi(0) & \varphi(0)\\ \psi'(0) & \varphi'(0)\end{vmatrix} = \begin{vmatrix}\psi(0) & 1\\ \psi'(0) & h\end{vmatrix} = -B_0(\psi)$$

$$= \begin{vmatrix}\psi(\pi) & \varphi(\pi)\\ \psi'(\pi) & \varphi'(\pi)\end{vmatrix} = \begin{vmatrix}1 & \varphi(\pi)\\ -H & \varphi'(\pi)\end{vmatrix} = B_1(\varphi)$$

$\Delta(\lambda)$ is an <u>entire</u> function.

Roots of $\Delta(\lambda)$ are the eigenvalues of (L,h,H).

- The set of evals of (L,h,H) is discrete w/ no acc. pts.
- The eigenvalues $\{\lambda_i\}_{i=1}^{\infty}$ are real
- The e.space is spanned by $\psi(x,\lambda_n) = \beta_n\varphi(x,\lambda_n)$
- $\int_0^\pi \varphi(x,\lambda_n)\varphi(x,\lambda_m)\,dx = \delta_{nm}\alpha_n$
- $\beta_n\alpha_n = -\frac{d\Delta}{d\lambda}(\lambda_n)$

- Case $q=0$:

$$c(x,\lambda) = \cos\rho x \qquad\qquad \rho^2=\lambda$$
$$s(x,\lambda) = \rho^{-1}\sin\rho x$$
$$\varphi(x,\lambda) = \cos\rho x + h\rho^{-1}\sin\rho x$$
$$\psi(x,\lambda) = \cos\rho(\pi-x) + H\rho^{-1}\sin\rho(\pi-x)$$

Notice that $\cos\rho x = \frac{1}{2}(e^{i\rho x}+e^{-i\rho x})$, so $\cos\rho x = O(e^{|\operatorname{Im}\rho|x})$

$\rho^{-1}\sin\rho x = \frac{1}{2\rho}(e^{i\rho x}-e^{-i\rho x})$, so $\rho^{-1}\sin\rho x = O(\frac{1}{|\rho|}e^{|\operatorname{Im}\rho|x})$

$\rho\cos\rho x = O(|\rho|e^{|\operatorname{Im}\rho|x})$ ...

- Case general q — compare with $q=0$.

$$c(x,\lambda) = \cos\rho x + \int_0^x \rho^{-1}\sin\rho(x-y)q(y)c(y,\lambda)\,dy$$

[b/c $-c''-\lambda c = -q(x)c$ and $c(0,\lambda)=1$, $c'(0,\lambda)=0$]

$$s(x,\lambda) = \rho^{-1}\sin\rho x + \int_0^x \rho^{-1}\sin\rho(x-y)q(y)s(y,\lambda)\,dy$$

Also, $\varphi(x,\lambda) = c(x,\lambda)+hs(x,\lambda)$

$$= \cos\rho x + \rho^{-1}\sin\rho x + \int_0^x \rho^{-1}\sin\rho(x-y)\underbrace{q(y)}_{\text{"forcing"}}\varphi(y,\lambda)\,dy$$