

Some notes on 1D systems and reductions

Ref: Steven Strogatz
Nonlinear Dynamics
and Chaos

For an ODE $\dot{x} = F(x)$ in \mathbb{R}^n , if $F(x) = -\nabla V(x)$ for some scalar potential function V , then

$$\frac{dV(x(t))}{dt} = \nabla V \cdot \dot{x} = -|\nabla V|^2 \leq 0$$

so V is decreasing along flows unless the flow is a fixed point.

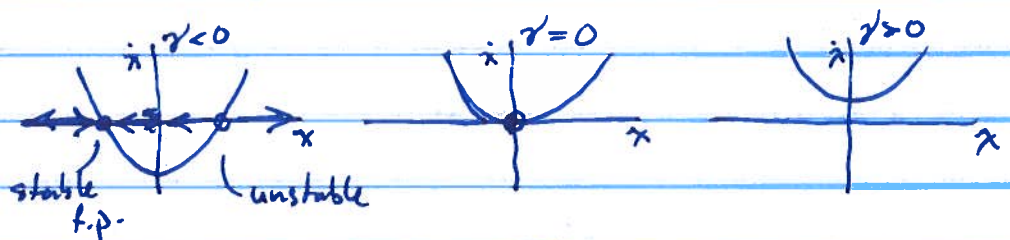
In 1D, such a potential $V(x)$ always exists. This makes it analytically easy to prove that periodic solutions (besides fixed points) do not exist.

1D systems become more interesting when a parameter is involved. As the parameter varies, the fixed points of the system and their stability changes, resulting in bifurcations.

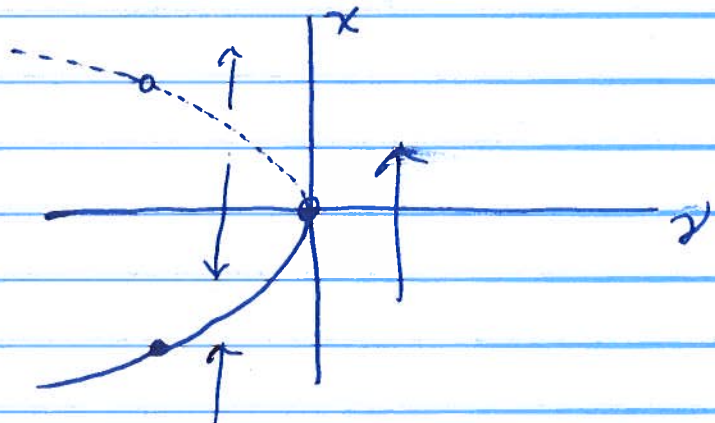
- Saddle-node bifurcation, or blue-sky bifurcation.

typical form: $\dot{x} = x^2 + \gamma$

phase-line analysis:



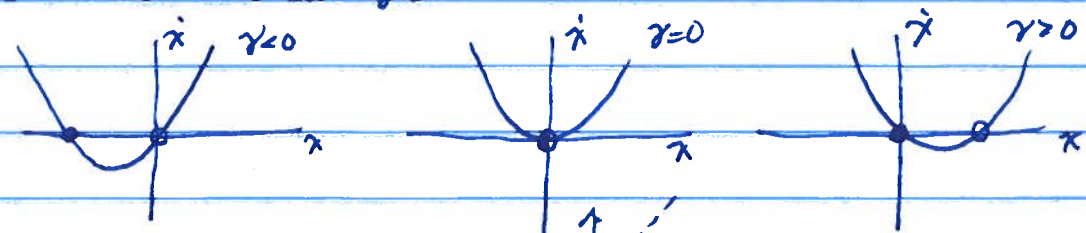
bifurcation diagram:



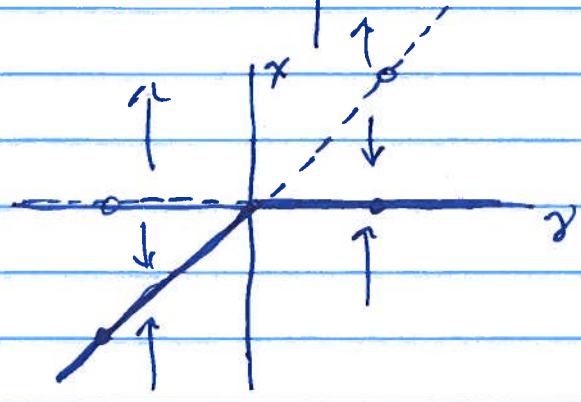
transcritical bifurcation

typical form: $\dot{x} = x(x - \gamma)$

phase-line analysis:



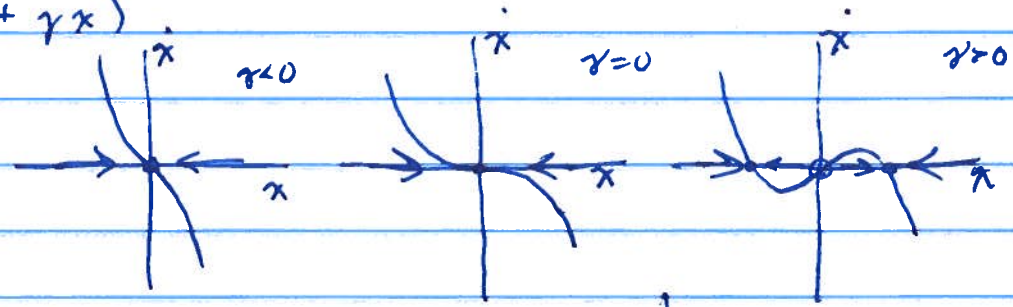
bifurcation diagram:



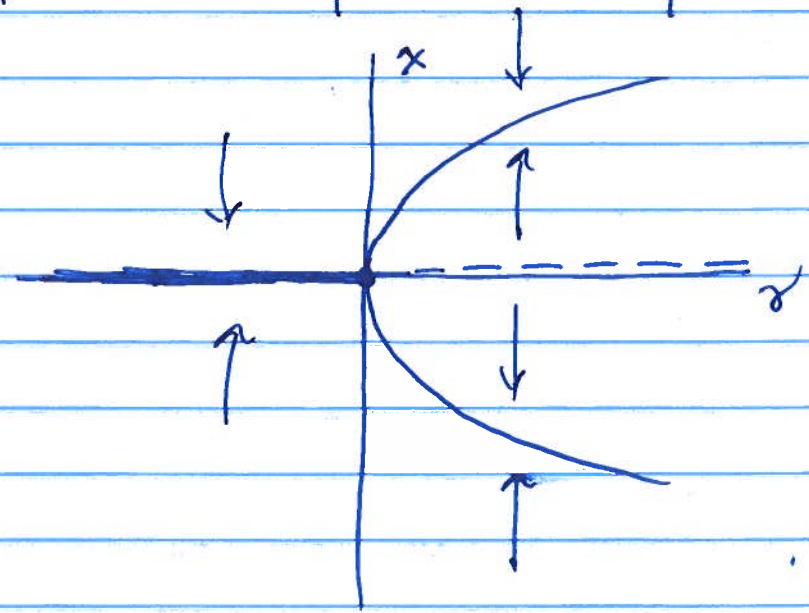
pitchfork bifurcation

$\dot{x} = -(x^3 + \gamma x)$

phase-line analysis:



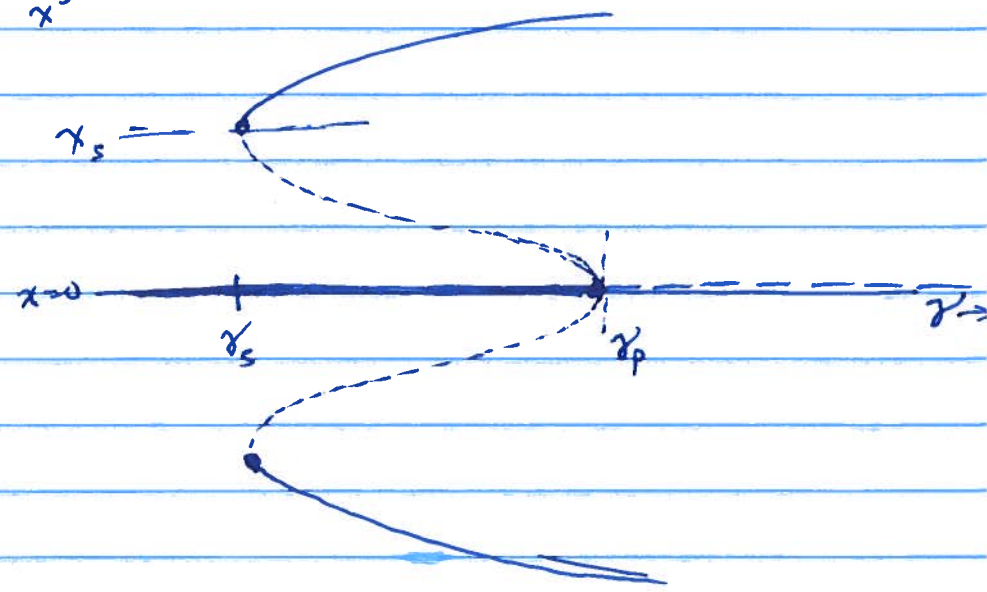
bifurcation diagram:



- Combination of blue-sky and pitchfork bifurcations

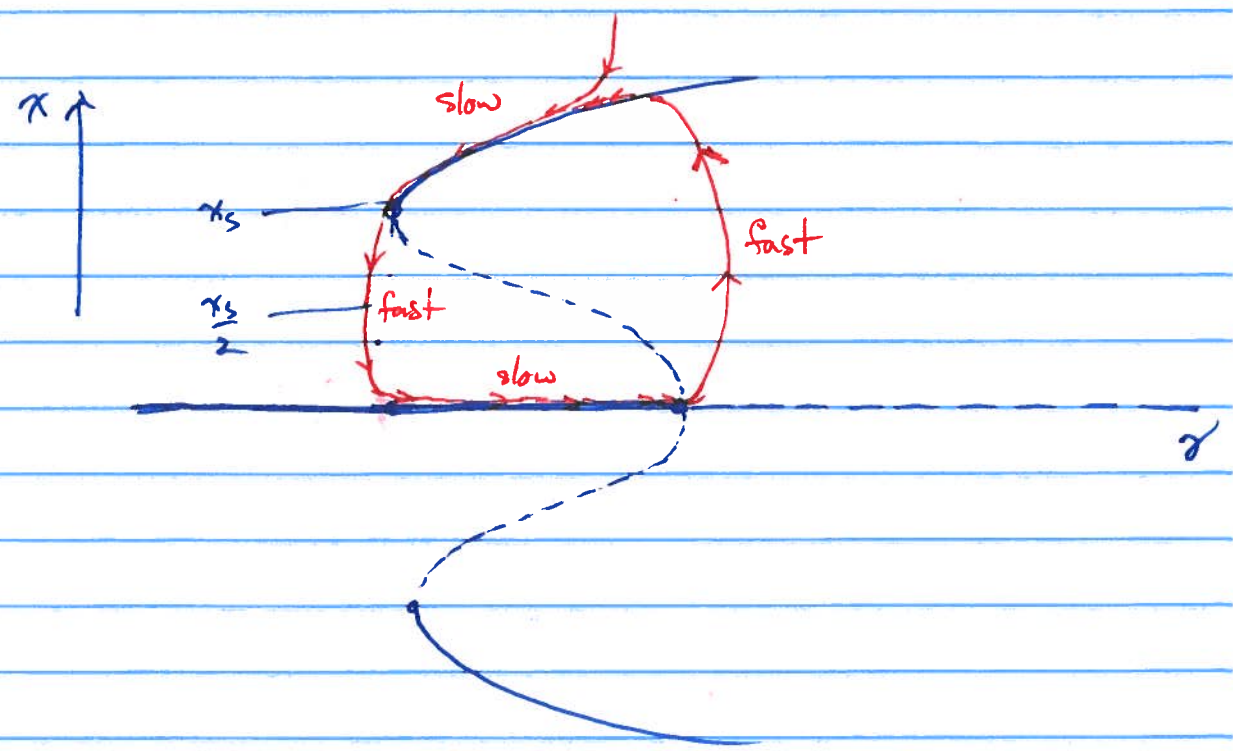
$$\dot{x} = \gamma x + x^3 - x^5$$

bifurcation diagram:



- Hysteresis

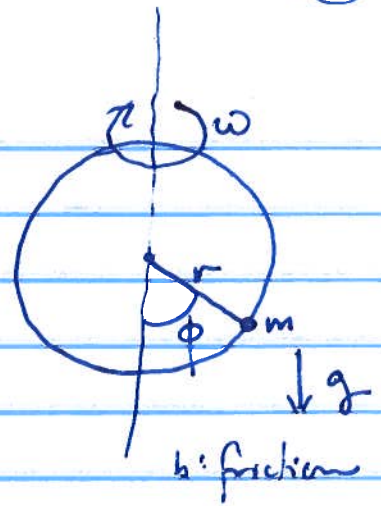
$$\begin{aligned} \dot{x} &= \gamma x + x^3 - x^5 \\ \dot{\gamma} &= \epsilon(4x^2 - \gamma^2) \end{aligned}$$



- Overdamped bead on a rotating loop.

Second-order eqn. for ϕ :

$$mr\ddot{\phi} = -b\dot{\phi} - mg\sin\phi + mr\omega^2\sin\phi\cos\phi$$



Non-dimensionalization:

- First divide by the force mg to make each term ~~realistic~~ have units of 'time and set $t = T\tau$ where T is a fixed unit of time and $\tau \in \mathbb{R}$.

(so replace $\dot{\phi} = \frac{d\phi}{dt} = \frac{1}{T} \frac{d\phi}{d\tau}$, $\ddot{\phi} = \frac{1}{T^2} \frac{d^2\phi}{d\tau^2}$):

$$\left(\frac{r}{gT^2}\right) \frac{d^2\phi}{d\tau^2} = -\left(\frac{b}{mgT}\right) \frac{d\phi}{d\tau} - \sin\phi + \left(\frac{r\omega^2}{g}\right) \sin\phi\cos\phi$$

- The "overdamped regime" is when $\frac{r}{gT^2}$ is very small ~~and~~ compared with $\frac{b}{mgT}$.

So let's set $\frac{b}{mgT} = 1$ so that $T = \frac{b}{mg}$

and set $\varepsilon = \frac{r}{gT^2} = \frac{m^2 g r}{b^2}$, $\gamma = \frac{r\omega^2}{g}$

$$\Rightarrow \varepsilon \frac{d^2\phi}{d\tau^2} = -\frac{d\phi}{d\tau} - \sin\phi + \gamma \sin\phi\cos\phi$$

- Now brutally remove the second-order derivative to get a first-order system (RHS = 0). This system experiences a blue sky bifurcation: As γ increases (say, you increase the rotation speed), the bead moves from $\phi = 0$ to $\phi = \phi_s$.

Two-dimensional systems and the Poincaré map.

example The Lotka-Volterra model with a prey source (LV)

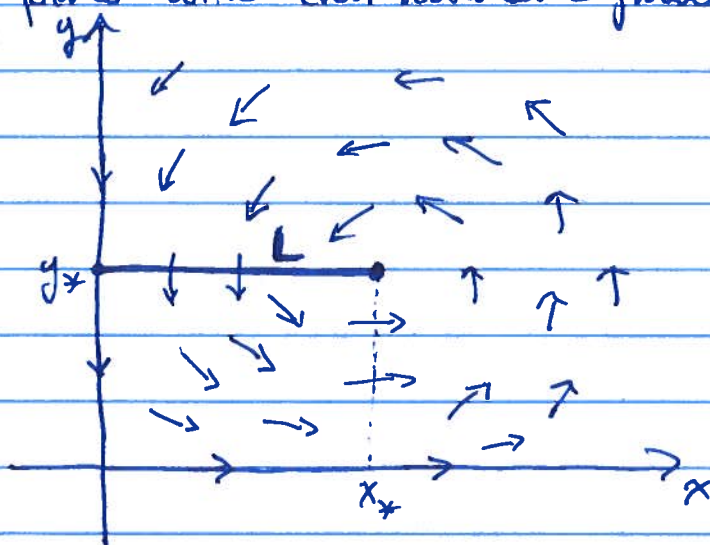
$$\dot{x} = kx - axy + sy$$

$$\dot{y} = -ly + bxy$$

The equilibrium points are $\begin{cases} (x,y) = (0,0) \\ (x,y) = \left(\frac{l}{b}, \frac{k}{a - s\frac{l}{b}}\right) \end{cases}$

More generally, consider a similar system with the coordinate axes as isoclines and with equilibrium points $(x,y) = (0,0)$ and $(x,y) = (x_*, y_*)$

Typical phase plane with cross-sectional segment $L = (x_*, y_*)$



Stability of the LV model with prey source :

$$\frac{\partial(\dot{x}, \dot{y})}{\partial(x, y)} = \begin{bmatrix} k - ay & -ax + s \\ by & -l + bx \end{bmatrix}$$

$$\frac{\partial(\dot{x}, \dot{y})}{\partial(x, y)} \Big|_{(x^*, y^*)} = \begin{bmatrix} k \frac{sb}{sb - la} & \frac{1}{b}(sb - la) \\ k \frac{-lb}{sb - la} & 0 \end{bmatrix}$$

Notice that the real part of the eigenvalues, when $s \geq 0$ and s is sufficiently small is negative (and the two evals are complex conjugates). But when $s = 0$, the eigenvalues are purely imaginary. In the first case, the fixed point is stable. In the second case, the linearization does not tell us about stability (higher order derivatives of $F(x, y) = (\dot{x}, \dot{y})$ would).

We can show that, for $s = 0$, the system has a continuum of periodic trajectories as follows.

The ODE system is

$$(\dot{x}, \dot{y}) = F(x, y) = (x(k - ay), y(-l + bx))$$

Consider the scaled rotated vector field

$$\frac{1}{xy} F^\perp(x, y) = \left(\frac{l - bx}{x}, \frac{k - ay}{y} \right),$$

and observe that it is the gradient of

$$g(x,y) = \int^x \frac{d-bx'}{x'} dx' + \int^y \frac{k-ay'}{y'} dy'$$

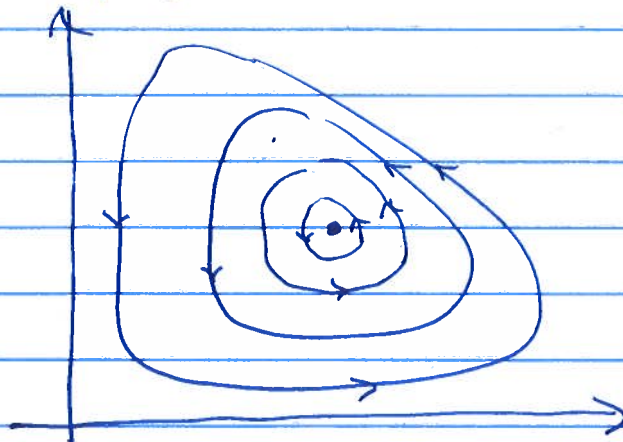
Thus ~~the~~ ^{any} solution of $(\dot{x}, \dot{y}) = F(x,y)$ satisfies

$$\frac{d}{dt} g(x(t), y(t)) = \nabla g(x(t), y(t)) \cdot (\dot{x}(t), \dot{y}(t))$$

$$= \frac{1}{xy} F^\perp(x,y) \cdot F(x,y) = 0$$

Thus g is constant on the trajectories of the system, so the level sets of g are the trajectories
 connected components of $\{g=c\}$

The level sets of g in this case are closed curves and the point (x_0, y_0) :



For systems of this general ilk, one can ask the following questions:

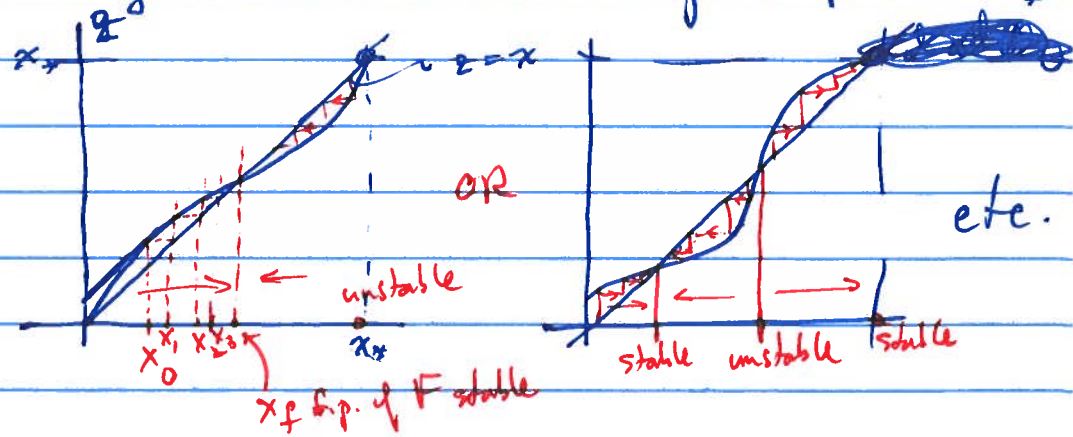
(1) Is it reasonable that, (for appropriate parameters) one could prove that any trajectory that begins on L will return to L .

(2) Given that (1) holds, let (x_0, y_0) be an initial condition, let (x_1, y_1) be the first return of the solution to L , (x_2, y_2) the second return, ..., (x_n, y_n) the n^{th} return. This generates a discrete dynamical system

$$x_{n+1} = F(x_n),$$

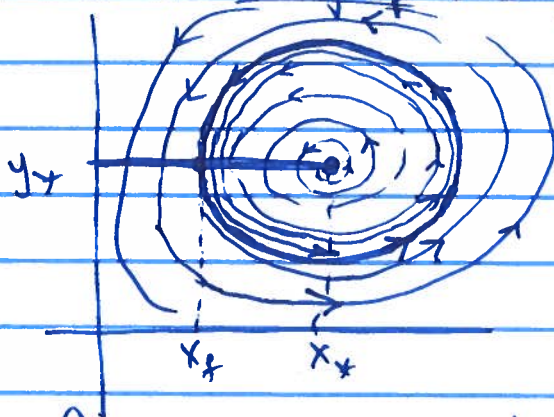
in which F is called the return map, or the Poincaré map (for this ODE and the segment L).

(3) One can prove (this is unique to 2D) that F is strictly increasing and continuous with a fixed point at x_* .



Thus, the Poincaré map is a discrete dynamical system that is easily visualized graphically by a "cobweb", or "Lamerey" diagram.

(4) Fixed points of F ($F(x_f) = x_f$) correspond to periodic orbits of the ODE. The stability of x_f matches the stability of the periodic orbit.



In this figure, x_f is unstable for F , so (x_f, y_f) is ^{repelling} unstable for the ODE. x_g is stable for F , so the periodic orbit through (x_g, y_g) is stable, or attracting for the ODE.

Question: When does an ODE system in 2D admit a Poincaré map? Sometimes the Poincaré-Bendixson Theorem can be applied to address this question.

Defn (5) First we define the ω -limit (omega-limit) $\omega[x]$ of an orbit γ [$\gamma(t) = (x(t), y(t))$ solution of $(\dot{x}, \dot{y}) = F(x, y)$]. $\omega[x]$ is the set of all points $p = (x, y)$ such that there exists a sequence $\{t_n\}_{n=1}^{\infty}$ of increasing times leading to ∞ such that $\gamma(t_n) \rightarrow p$ as $n \rightarrow \infty$.

(k) The Poincaré-Bendixson Theorem

Let $\dot{x} = F(x)$ be an ODE in \mathbb{R}^2 with F continuously differentiable, and let γ be an orbit thereof. Then $\omega[\gamma]$ has one of the following properties:

- $\omega[\gamma]$ is empty
- $\omega[\gamma]$ contains infinitely many fixed points
- $\omega[\gamma]$ consists of a single fixed point
- $\omega[\gamma]$ is a periodic orbit
- $\omega[\gamma]$ consists of finitely many fixed points connected by homoclinic or heteroclinic orbits.