Math 7320 @ LSU
Spring, 2018
Problem Set 1

Textbook problems refer to section number, subsection number, and problem number in V.I. Arnold's book "Ordinary Differential Equations".

1. Prove that $e^{A+B} = e^A e^B$ if $AB = BA$; and prove that

$$e^{t(A+B)} = e^{tA}\, e^{tB} \qquad \text{for all } t \in \mathbb{R}$$

if and only if $AB = BA$.

2. Verify the details of the computations for $e^{t\Lambda}$ at the bottom of page 6 of the lecture notes titled "General ideas of flows...".

3. Prove that the differentiation operator d/dx in the space of polynomials $p(x)$ of degree less than a fixed positive integer n is nilpotent.

4. Let λ be a complex number. A *quasi-polynomial with exponent* λ is a product $e^{\lambda x}p(x)$, where $p(x)$ is a polynomial. The degree of $p(x)$ is the degree of the quasi-polynomial $e^{\lambda x}p(x)$. Let λ and μ be two distinct complex numbers and n a positive integer. Consider the vector space of functions generated by the quasi-polynomials with exponent λ or μ and degree less than n. Find its dimension.

5. Prove that every function $f : \mathbb{R} \to \mathbb{C}$ that can be written in the form of a quasi-polynomial has a unique expression of this form.

6. (Theorem in §14.9 of Arnol'd: *Ordinary Differential Equations.*) Let P_n denote the vector space of polynomials of degree less than n. The differentiation operator d/dx is a linear operator from P_n to P_n, considered as functions from $\mathbb{R}$ to $\mathbb{C}$, and, for any $t \in \mathbb{R}$,

$$e^{t\frac{d}{dx}} = H^t, \tag{1}$$

where H^t is the shift operator, that is, $(H^t f)(x) = f(x+t)$.
Prove this theorem (or see Arnol'd's proof). Then prove that this theorem is valid on larger functions spaces. How broad can you make the space of functions on which $\exp(t\frac{d}{dx}) = H^t$ holds? (Interpret $\exp(t\frac{d}{dx})$ as the solution operator for the appropriate differential equation.)

7. Prove that

$$\det e^A = e^{\operatorname{tr} A}$$

by using normal forms. Here, A is a linear operator in a finite-dimensional vector space over $\mathbb{R}$ or over $\mathbb{C}$.

8. Let A be an operator in a complex vector space of dimension n, and let ${}^{\mathbb{R}}A$ be the realification of A, acting on a real vector space of dimension $2n$. Prove that

$$\det e^{t\,{}^{\mathbb{R}}A} = \prod_{i=1}^{n} e^{2t\,\mathrm{Re}\lambda_i},$$

in which the λ_i are the eigenvalues of A (counted with multiplicity).