Math 7320 @ LSU
Spring, 2019
Problem Set 2

Textbook problems refer to section number, subsection number, and problem number in V.I. Arnold's book "Ordinary Differential Equations".

**1.** This problem describes Putzer's algorithm for computing the solution of a linear constant-coefficient homogeneous system $\dot{x} = Ax$ without first computing the Jordan normal form for the generator $A$. Let $A$ be a linear operator in a finite-dimensional complex vector space of dimension $n$, and let $\{\lambda_i\}_{i=1}^n$ denotes its eigenvalues. For $k = 1, \ldots, n$, define the operators

$$A_k = \prod_{j=1}^{k} (A - \lambda_j E)$$

and the scalar functions $r_j(t)$ as the solution of the system

$$\begin{aligned} dr_1/dt &= \lambda_1 r_1\,, & r_1(0) &= 1 \\ dr_j/dt &= \lambda_j r_j + r_{j-1}\,, & r_j(0) &= 0 \quad \text{for } j \geq 2. \end{aligned}$$

Putzer's theorem states that

$$e^{tA} = \sum_{k=0}^{n-1} r_{k+1}(t) A_k.$$

**2.** Consider the linear non-autonomous nonhomogeneous linear $n^{th}$-order initial-value ordinary differential equation

$$\begin{aligned} & x^{(n)}(t) + \sum_{\ell=1}^{n} a_\ell(t)\, x^{(n-\ell)}(t) = F(t) \\ & x^{(\ell)}(t_0) = c_\ell\,, \quad (0 \leq \ell < n) \end{aligned} \tag{1}$$

Set $\phi(t) = x^{(n)}(t)$. Prove that, if $x(t)$ satisfies (1), then $\phi(t)$ satisfies the Volterra integral equation

$$\phi(t) + \int_{t_0}^{t} K(t,s)\phi(s) \;=\; f(t)\,,$$

in which the kernel $K$ is defined by

$$K(t,s) \;=\; \sum_{\ell=1}^{n} a_\ell(t) \frac{(t-s)^{\ell-1}}{(\ell-1)!}$$

and the term of inhomogeneity is

$$f(t) = F(t) - \sum_{\ell=1}^{n} a_\ell(t) \left( c_{n-1} \frac{(t-t_0)^{\ell-1}}{(\ell-1)!} + \cdots + c_{n-\ell} \right).$$

**3.** Consider the spectral Volterra-integral problem

$$\phi(t) \;-\; \lambda \int_{t_0}^{t} K(t,s)\phi(s)\,ds \;=\; f(t) \qquad (t_0 \le t \le t_1),$$

in the $L^2$ theory. Prove that, for fixed $f$, the solution $\phi$ is analytic in $\lambda$ and satisfies a Volterra integral equation

$$\phi(t) \;=\; f(t) - \lambda \int_{t_0}^{t} H(t,s;\lambda) f(s)\,ds \qquad (t_0 \le t \le t_1).$$

Find the kernel $H(t,s;\lambda)$ in terms of the kernel $K(t,s)$.