Math 7320 @ LSU
Spring, 2019
Problem Set 4

1. Let J be a finite interval in the real line, and let B be the ball $B = \{y \in \mathbb{R}^n \mid |y - x_0| \leq \rho\}$, where $x_0 \in \mathbb{R}^n$ and $\rho > 0$ are fixed. Prove that the set of functions

$$X = \{x : J \to B \mid x \text{ is continuous}\}$$

endowed with the supremum norm

$$\|x_1 - x_2\|_{\sup} = \sup_{t \in J} |x_1(t) - x_2(t)|$$

is a complete metric space.

2. Let S be a subset of $\mathbb{R}^n$. Prove the following. If a function $F : S \to \mathbb{R}^m$ is of Lipschitz class, then F is continuous, and, if S is also bounded, then the range of F is bounded.

3. Let X be defined as in problem (1), let t_0 be in J, and let $F : B \to \mathbb{R}^n$ be of Lipschitz class. Prove that, for each function $x \in X$, the following two conditions are equivalent.

(IVP) x is differentiable at each point of J, $\dot{x} = F(x)$, and $x(t_0) = x_0$.

(IE) $x(t) = x_0 + \int_{t_0}^{t} F(x(s))\, ds$.

4. Let $F : B \to \mathbb{R}^n$ be a Lipschitz function and p a positive number. If $x : (a, b) \to B$ is a solution to $\dot{x} = F(x)$ and $x(t_0) = x(t_0 + p)$ for some $t_0 \in (a, b)$ and some $p \in (a, b) - t_0$, then x can be extended uniquely to a periodic function $x : \mathbb{R} \to B$ that satisfies $\dot{x} = F(x)$ and $x(t + p) = x(t)$ for all $t \in \mathbb{R}$.

5. Prove the following theorem.
Let $F : (t_0 - b, t_0 + b) \times B \to \mathbb{R}^n$ be Lipschitz continuous with constant K, and let F be bounded by M. Then there is an interval $t_0 + (-a, a)$ such that the initial-value problem

$$\begin{cases} \dot{x} = F(t, x) \\ x(t_0) = x_0 \end{cases}$$

admits a unique solution $x : t_0 + (-a, a) \to B$.

6. Prove that the function $R(s, \xi)$ at the end of page 9 of the class notes on "Existence and Uniqueness" converges uniformly over s to 0 as $x(s, \xi) - x(s, 0) \to 0$.