Math 7320 @ LSU
Spring, 2019
Problem Set 5

1. (From S. Strogatz, "Nonlinear Dynamics and Chaos", Perseus Publishing, 2000, p. 86) Consider the 1D system

$$\dot{x} = h + rx - x^2.$$

When $h = 0$, this system undergoes a transcritical bifurcation in the parameter r at $r = 0$. The goal is to see how the bifurcation diagram of fixed points vs. r is affected by the "imperfection parameter" h.

a. Plot the bifurcation diagram for $\dot{x} = h + rx - x^2$, for $h < 0$, $h = 0$, and $h > 0$.

b. Sketch the regions in the (r, h) plane that correspond to qualitatively different vector fields, and identify the bifurcations that occur on the boundaries of those regions.

c. Plot the potential $V(x)$ corresponding to all the different regions in the (r, h) plane.

2. Consider the system

$$\begin{aligned} \dot{x} &= -x(y + x^2 - 2x - 1) \\ \dot{y} &= y(x - 1) \end{aligned}$$

a. Prove that this system admits a Poincaré recurrence map on the horizontal line segment between the point $(0, 2)$ and the point $(1, 2)$.

b. Prove that the x-coordinate of this recurrence map is a non-decreasing function.

3. (From S. Strogatz) Consider the system

$$\begin{aligned} \dot{x} &= x - y - x(x^2 + 5\,y^2) \\ \dot{y} &= x + y - y(x^2 + y^2) \end{aligned}$$

a. Classify the fixed point at the origin according to the structure of the solutions of its linearization.

b. Prove that the system has a periodic orbit. It is convenient to do this by first converting the system to polar coordinates, using $r\dot{r} = x\dot{x} + y\dot{y}$ and $\dot{\theta} = (x\dot{y} - y\dot{x})/r^2$.

(Continued on the next page.)

4. Consider the discrete dynamical system in the interval $[0, 1]$ defined by iterating the "tent-map"

$$F(x) \;=\; 1 - |2x - 1|\,,$$

that is, with $x_0 \in [0, 1]$ given, a sequence $\{x_n\}_{n=0}^{\infty}$ in $[0, 1]$ is determined by

$$x_{n+1} = F(x_n)\,.$$

For each positive integer q, determine how many cycles (periodic orbits) of period q this system has. Prove that periodic orbits are dense in $[0, 1]$.