Local existence and uniqueness theory for the nonlinear initial-value problem

$$(\mathrm{IVP}) \quad \begin{cases} \dot{x} = F(x) \\ x(t_0) = x_0 \end{cases} .$$

Given positive numbers a and ρ, solutions of this IVP will be considered in the space of continuous functions $x : J_a \to B_\rho$, in which

$$J_a = t_0 + (-a, a)$$
$$B_\rho = \{ y \in \mathbb{R}^n : |y - x_0| \le \rho \} .$$

This set will be denoted by X :

$$X = \{ x : J_a \to B_\rho : x \text{ is continuous} \} .$$

When endowed with the following notion of distance,

$$\| x_1 - x_2 \|_{\sup} = \sup_{t \in J_a} | x_1(t) - x_2(t) | ,$$

X becomes a complete metric space. You can prove this by the following steps (<u>exercise</u>):

(1) The linear space $BC(J_a, \mathbb{R}^n)$ of bounded continuous functions from J_a to $\mathbb{R}^n$, with the sup norm $\|x\|_{\sup} = \sup_{t \in J_a} |x(t)|$ is a Banach space (closed normed vector space).

(2) $\|x - y\|_{\sup}$ defines a metric on any subset of BC.

(3) B_ρ being closed (and bounded) makes X complete in this metric.

<u>Defn</u> For a subset $S \subset \mathbb{R}^n$, a function $F: S \to \mathbb{R}^n$ is Lipschitz with Lipschitz constant $K > 0$ if $\forall\, y_1, y_2 \in S$, $|F(y_1) - F(y_2)| \le K|y_1 - y_2|$.

<u>Exercise</u> If $F: S \to \mathbb{R}^n$ is Lipschitz, then F is continuous. If S is bounded and $F: S \to \mathbb{R}^n$ is Lipschitz, then F is bounded.

<u>Theorem</u> Let $F: B_\rho \to \mathbb{R}^n$ be Lipschitz with constant K, and set $M = \sup_{y \in B_\rho} |F(y)|$. If $a > 0$ is such that $a < \min\{\frac{\rho}{M}, \frac{1}{K}\}$, then there exists a unique function $x \in X$ that satisfies IVP.

<u>Proof</u> First, prove that for any $x \in X$, the following two conditions are equivalent

$$(\text{IVP}) \quad \dot{x} = F(x) \quad \text{and} \quad x(t_0) = x_0$$

$$(\text{IE}) \quad x(t) = x_0 + \int_{t_0}^{t} F(x(s))\, ds,$$

in which (IVP) stands for "initial-value problem" and (IE) stands for "integral equation". (In (IVP), it is tacitly assumed that the differentiability of x is part of the condition.) Details are left as an exercise.

We will prove that (IE) admits one and only one solution $x \in X$.

Defn: A function $\mathcal{F}: X \to X$ is a contraction if $\exists\, r \in \mathbb{R}$ such that $0 < r < 1$ so that $\forall\, x_1, x_2 \in X$, one has

$$\|\mathcal{F}x_1 - \mathcal{F}x_2\| \leq r\|x_1 - x_2\|.$$

We will prove now that the map

$$(\mathcal{F}x)(t) := x_0 + \int_{t_0}^{t} F(x(s))\,ds \qquad (t \in J_a)$$

defines a contraction on X: Given a continuous function $x: J_a \to B_\rho$, $F \circ x : J_a \to \mathbb{R}^n$ is also continuous so that $(\mathcal{F}x)(t)$ is well defined and

$$|\mathcal{F}x(t) - x_0| \leq \int_{t_0}^{t} |F(x(s))|\,ds \leq aM < \rho$$

so that $\mathcal{F}x \in X$. ($\mathcal{F}x$ is continuous because $\int_{t_0}^{t} F(x(s))ds$ is.)

For $x_1, x_2 \in X$,

$$(\mathcal{F}x_1 - \mathcal{F}x_2)(t) = \int_{t_0}^{t} [F(x_1(s)) - F(x_2(s))]\,ds,$$

and thus, by the Lipschitz property of F, $\forall\, t \in J_a$,

$$|(\mathcal{F}x_1 - \mathcal{F}x_2)(t)| \leq \int_{t_0}^{t} |F(x_1(s)) - F(x_2(s))|\,ds$$

$$\leq \int_{t_0}^{t} K|x_1(s) - x_2(s)| \leq aK\|x_1 - x_2\|_{sup}$$

and thus $\|\mathcal{F}x_1 - \mathcal{F}x_2\|_{sup} \leq aK\|x_1 - x_2\|_{sup}$. ~~w/ aK < 1,~~

Since $aK < 1$, $\mathcal{F}$ is a contraction in X.

Notice also that $\forall\, x \in X$, $(\mathcal{F}x)(t_0) = x_0$.

Now we see that (IUP) and (IE) are equivalent to

(FP) $\mathcal{F} : X \to X$ admits a unique fixed point.

Indeed, (IE) is exactly the statement that $\mathcal{F}x = x$.

To prove existence, choose $y_0 \in X$ with $y_0(t_0) = x_0$ and define recursively $y_{n+1} = \mathcal{F} y_n$ for $n \geq 1$, [that is, $y_{n+1}(t) = x_0 + \int_{t_0}^{t} F(y_n(s))\, ds$].

Inductively, one proves by the contraction property of $\mathcal{F}$, that

$$\| y_{n+1} - y_n \|_{\sup} \leq (aK)^n \| y_1 - y_0 \|_{\sup} .$$

Since $aK < 1$, the exponential convergence of $\| y_{n+1} - y_n \|_{\sup}$ to zero makes $\{y_n\}_{n=0}^{\infty}$ a Cauchy sequence in X. Since X is complete, this sequence admits a limit ~~point~~ function $x \in X$, and it is a fixed point of $\mathcal{F}$ because

$$\mathcal{F}x = \mathcal{F} \lim_{n\to\infty} y_n = \lim_{n\to\infty} \mathcal{F} y_n = \lim y_{n+1} = x .$$

A contraction has but a unique fixed point because if $\mathcal{F}x_1 = x_1$ and $\mathcal{F}x_2 = x_2$, then

$$\| x_1 - x_2 \| = \| \mathcal{F}x_1 - \mathcal{F}x_2 \| \leq aK \| x_1 - x_2 \| ,$$

and since $aK < 1$, one has $\| x_1 - x_2 \| = 0$, or $x_1 = x_2$.

Some consequences of the existence/uniqueness theorem:

- Given a solution $x : t_0 + (-a,a) \to B_\rho$ with $x(t_0) = x_0$, the function $y : t_1 + (-a,a) \to B_\rho$ defined by

  $$y(t) = x(t + t_0 - t_1)$$

  satisfies $\dot{y} = F(y)$, $y(t_1) = x_0$.

  We say that the system is time-shift-invariant, and it is called "autonomous". This comes from the fact that F is a function of x and not t.

- Solutions can't cross:

  If $\dot{x} = F(x)$ and $\dot{y} = F(y)$ and $x(t_0) = x_0 = y(t_1)$, then $y(t_1 + s) = x(t_0 + s)$ on the maximal s-interval on which both are defined.

- Periodic solutions:

  If $x : (a,b) \to B_\rho$ is a solution to $\dot{x} = F(x)$ and $x(t_0) = x(t_0 + p)$ for some $t_0 \in (a,b)$ and p such that $t_0 + p \in (a,b)$, then x can be extended uniquely to a periodic function $x : \mathbb{R} \to B_\rho$ that satisfies $\dot{x} = F(x)$ and $x(t+p) = x(t)$ for all $t \in \mathbb{R}$. [You can do the details.]

<u>Exercise</u> Nonautonomous systems.

Let $F: t_0 + (-b, b) \times B_\rho \longrightarrow \mathbb{R}^n$ be Lipschitz continuous with constant K, and let F be bounded by M. Then there is an interval $t_0 + (-a, a)$ such that the IVP

$$\begin{cases} \dot{x} = F(t, x) \\ x(t_0) = x_0 \end{cases}$$

admits a unique solution $x: t_0 + (-a, a) \longrightarrow B_\rho$.

You can ~~do~~ prove this by applying the theorem for autonomous systems to the extended autonomous system

$$\begin{cases} \dot{x} = F(x, t) \\ \dot{t} = 1 \end{cases}.$$

To show that $a < \min\left\{\frac{1}{K}, \frac{1}{M}, b\right\}$ works, you have to do just a little bit more than directly applying the theorem.

Lemma (Gronwall's inequality)

Let $f : [0, \alpha] \longrightarrow \mathbb{R}$ be a continuous function with non-negative values, and let $C, K \geq 0$ be such that

$$f(t) \leq C + \int_0^t K f(s)\, ds \qquad \forall \; t \in [0, \alpha].$$

Then $f(t) \leq Ce^{Kt}$.

Proof For $C > 0$; set $g(t) := C + \int_0^t K f(s)\, ds$.

Then $g'(t) = K f(t)$, so that $\frac{g'(t)}{g(t)} \leq K$.

Thus $\ln g(t) = \ln g(0) + \int_0^t \frac{d}{ds} \ln g(s)\, ds \leq \ln C + Kt$

and so $g(t) \leq Ce^{Kt}$, and $\therefore$ $f(t) \leq Ce^{Kt}$.

For $C = 0$, one has $f(t) \leq C + \int_0^t K f(s)\, ds \;\; \forall \; C > 0$,

and thus $f(t) \leq Ce^{Kt} \;\; \forall \; C > 0$. This implies

$f(t) \leq 0 e^{Kt}$ ∎

Theorem (Continuous dependence on initial conditions)

Let x_1 and x_2 be solutions to the ODE $\dot{x} = F(x)$ on the interval $(-a, a)+t_0$. Then

$$|x_1(t) - x_2(t)| \leq |x_1(t_0) - x_2(t_0)|\, e^{K|t - t_0|}$$

for all $t \in (-a, a) + t_0$, where K is the Lipschitz const. for F.

<u>Proof</u> Apply Gronwall's inequality to $|x_1(t) - x_2(t)|$:

$$x_1(t) = x_1(t_0) + \int_{t_0}^{t} F(x_1(s))\,ds$$

$$x_2(t) = x_2(t_0) + \int_{t_0}^{t} F(x_2(s))\,ds$$

$$\Longrightarrow \quad |x_1(t) - x_2(t)| = |x_1(t_0) - x_2(t_0)| \pm \int_{t_0}^{t} |F(x_1(s)) - F(x_2(s))|\,ds$$

with the + sign if $t \geq t_0$ and the minus sign if $t \leq t_0$. Since $|F(x_1(s)) - F(x_2(s))| \leq K|x_1(s) - x_2(s)|$, an application of Gronwall's inequality yields

$$|x_1(t) - x_2(t)| \leq |x_1(t_0) - x_2(t_0)|\, e^{K|t - t_0|}. \quad ■$$

<u>Continuous dependence on parameters</u>

Consider a system in $\mathbb{R}^{n+k}$:

$$(\dot{x}, \dot{y}) = F(x,y)$$
$$x(t_0) = x_0$$
$$y(t_0) = y_0$$

that is specialized in that $F(x,y) \in \mathbb{R}^n \subset \mathbb{R}^n \times \mathbb{R}^k$.

Thus $\dot{y} = 0$ and the variable y can be viewed as a fixed parameter in the dynamical system, and the system is equivalent to $\begin{cases} \dot{x} = F(x, y_0) \\ x(t_0) = x_0 \end{cases}$.

If $F(\cdot, y)$ is uniformly Lipschitz (same K for all y in some set), the theorem on continuous dependence on initial conditions becomes a theorem on continuous dependence on parameters.

Assume now that F is continuously differentiable.

The <u>flow</u> of the solution x as the initial condition is varied.

Let $x(t,\xi)$ satisfy

$$\begin{cases} \dot{x}(t,\xi) = F(x(t,\xi)) \\ x(0,\xi) = x_0 + \xi \end{cases}$$

on a common t-interval $(-a,a)$ for $|\xi| < \varepsilon > 0$, with $x(t,\xi) \in B_r$

Let $U(t,\xi)$ satisfy the ODE IVP ← linear!

$(D_\xi x)$
$$\begin{cases} \dot{U}(t,\xi) = DF_{x(t,0)}\, U(t,\xi) \\ U(0,\xi) = \xi \end{cases}$$

on the same t-interval for $|\xi| < \varepsilon$, [illegible]

Define $g(t,\xi) = \left| x(t,\xi) - x(t,0) - U(t,\xi) \right|$.

We will prove that

$$(*) \qquad \frac{g(t,\xi)}{|\xi|} \longrightarrow 0 \quad \text{as} \quad \xi \to 0 .$$

From the IVPs that $x(t,\xi)$ and $U(t,\xi)$ satisfy, we obtain

$$g(t,\xi) \le \int_0^t \left| F(x(s,\xi)) - F(x(s,0)) - DF_{x(s,0)}\, U(s,\xi) \right| ds .$$

By the differentiability of F, we have

$$F(x(s,\xi)) - F(x(s,0)) = DF_{x(s,0)}\left(x(s,\xi) - x(s,0)\right) + R(s,\xi)\left|x(s,\xi) - x(s,0)\right| ,$$

in which $R(s,\xi) \to 0$ as $x(s,\xi) - x(s,0) \to 0$.

In fact this convergence is uniform over s because DF is uniformly continuous on B_ρ (recall B_ρ is closed).

[I leave it as an exercise to prove this. You should use the mean-value theorem applied to $F(x(s,\xi)) - F(x(s,0))$, considered as the endpoint values of the argument of F along a line segment.]

We have already seen from the theorem on continuous dependence on initial conditions, that

$$|x(s,\xi) - x(s,0)| \le |\xi| e^{K|s|} \le |\xi| e^{aK}.$$

Thus $R(s,\xi) \to 0$ as $\xi \to 0$, uniformly in s.

Putting $\tilde{R}(s,\xi) = R(s,\xi) e^{K|s|}$, we obtain

$$g(t,\xi) \le \int_0^t \left| DF_{x(s,0)}(x(s,\xi) - x(s,0) - U(s,\xi)) \right| ds + |\xi| \int_0^t |\tilde{R}(s,\xi)| ds$$

with $\hat{R}(s,\xi) \to 0$ as $\xi \to 0$ uniformly in s so that

~~also~~ $\int_0^t |\tilde{R}(s,\xi)| ds \longrightarrow 0$ as $\xi \to 0$.

Let $|DF_y| \le N$ $\forall$ $y \in B_\rho$. Then (since F is C^1 on B_ρ)

$$g(t,\xi) \le \int_0^t N g(s,\xi) ds + |\xi| \int_0^t |\tilde{R}(s,\xi)| ds$$

By Gronwall's inequality, $g(t,\xi) \le |\xi| \int_0^t |\tilde{R}(s,\xi)| ds \, e^{Nt}$, so

$$\frac{g(t,\xi)}{|\xi|} \le e^{Nt} \int_0^t |\tilde{R}(s,\xi)| ds \longrightarrow 0 \text{ as } \xi \to 0.$$

Since the system $(D_\xi x)$ is linear, the map

$$\xi \mapsto U(t,\xi)$$

is linear, and it is, by virtue of $(*)$, equal to the derivative of the flow $x(t,\xi)$, as a function of ξ.

Notice that, if x_0 is a fixed point for the DE $\dot{x} = F(x)$, then $x(t,0) \equiv x_0$ and the system $(D_\xi x)$ is autonomous and linear, so

$$U(t,\xi) = \exp\left(t\, DF_{x_0}\right).$$