Basics of Lagrangian and Hamiltonian mechanics.

Let a class-C^1 Lagrangian function L be given:

$$L : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R} \; :: \; (q, x) \longmapsto L(q, x).$$

Let $x, y \in \mathbb{R}^n$ and $t > 0$ be given, and define the class $\mathcal{A}_{x,y,t}$ of admissible "trajectories", or curves,

$$\mathcal{A} = \mathcal{A}_{x,y,t} := \{ w : [0,t] \to \mathbb{R}^n : w \text{ is } C^2, w(0) = y, w(t) = x \}$$

and define the class $\mathcal{A}_{0,t}$ of variations

$$\mathcal{A}_0 = \mathcal{A}_{0,t} := \{ w : [0,t] \to \mathbb{R}^n : w \text{ is } C^2, w(0) = 0, w(t) = 0 \}$$

The variables q and x (sometimes called $\dot{q}$ and q) are known as velocity and position.

The action of L along a trajectory $w \in \mathcal{A}$ is

$$I[w] := \int_0^t L(\dot{w}(s), w(s))\, ds.$$

Problem Find a minimizing curve ~~*~~ γ of I, that is, find $\gamma \in \mathcal{A}$ such that

$$I[\gamma] = \min_{w \in \mathcal{A}} I[w].$$

__Theorem__ A minimizer $\gamma \in \mathcal{A}$ of I satisfies the Euler-Lagrange ODE

$$\frac{d}{ds} D_q L(\dot\gamma(s), \gamma(s)) = D_x L(\dot\gamma(s), \gamma(s))$$

__Proof__ Let $v \in \mathcal{A}_0$ be given and h be a real variable.

$$\frac{d}{dh} I[\gamma + hv] = \frac{d}{dh}\int_0^t L(\dot\gamma(s) + h\dot v(s), \gamma(s) + hv(s))\,ds$$

$$= \int_0^t \left[D_q L(\dot\gamma(s) + h\dot v(s), \gamma(s) + hv(s)) \cdot \dot v(s) + D_x L(\dot\gamma(s) + h\dot v(s), \gamma(s) + hv(s)) \cdot v(s)\right] ds$$

integration by parts
suppress s-dependence

$$= \int_0^t \left[-\frac{d}{ds} D_q L(\dot\gamma + h\dot v, \gamma + hv) + D_x L(\dot\gamma + h\dot v, \gamma + hv)\right] v\,ds$$

Since γ is a minimizer for I, this derivative vanishes at $h = 0$:

$$\int_0^t \left[-\frac{d}{ds} D_q L(\dot\gamma(s), \gamma(s)) + D_x L(\dot\gamma(s), \gamma(s))\right] \cdot v(s)\,ds = 0,$$

and since this holds for all $v \in \mathcal{A}_0$, one obtains

$$-\frac{d}{ds} D_q L(\dot\gamma(s), \gamma(s)) + D_x L(\dot\gamma(s), \gamma(s)) = 0$$

for all $s \in [0, t]$.

Observe that a minimizer of I always satisfies the E-L equations, whereas a solution of the E-L equations ~~may~~ might not be a minimizer of I. Such solutions are called <u>critical</u> trajectories for I.

For a solution $\gamma(s)$ of the E-L equations, define the generalized momentum along the trajectory γ as

$$p(s) = D_q L(\dot\gamma(s), \gamma(s)).$$

Notice that the function $D_q L$ can be applied to pairs (q, x) apart from any trajectory:

$$D_q L : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}^n :: (q, x) \longmapsto p.$$

<u>Big assumption</u> Suppose that, for each $x \in \mathbb{R}^n$, the equation $p = D_q L(q, x)$ can be uniquely solved for q as a C^1 function of p. That is, suppose that there exists a function of class C^1

$$Q : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}^n :: (p, x) \longmapsto q$$

such that

$$D_q L(Q(p, x), x) = p \quad \forall\ p \in \mathbb{R}^n.$$

In this situation, the Hamiltonian function H is defined:

$$H : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$$
$$H(p,x) = p \cdot Q(p,x) - L(Q(p,x), x)$$

Shorthand for this is $\quad H = pq - L$.

Example $\quad L(q,x) = \frac{1}{2}m|q|^2 - \phi(x)$

The E-L equations are $\quad m\ddot{x}(s) = -D\phi(x(s))$,

and the Hamiltonian function is $\quad H(p,x) = \frac{1}{2m}|p|^2 + \phi(x)$.

$$\text{Kinetic energy} = \frac{1}{2}m|q|^2 = \frac{1}{2m}|p|^2$$
$$\text{Potential energy} = \phi(x)$$

Theorem A solution $x(s)$ of the E-L equations, together with the momentum $p(s) = D_q L(\dot{x}(s), x(s))$, satisfy Hamilton's OD equations:

$$\dot{x}(s) = D_p H(p(s), x(s))$$
$$\dot{p}(s) = -D_x H(p(s), x(s)) \quad ,$$

and $H(p(s), x(s)) = \text{const}$, that is, H is ~~a~~ a conserved quantity for the system.

<u>Proof</u> For all $(p,x) \in \mathbb{R}^n \times \mathbb{R}^n$ and $\forall\ i = 1 \dots n$,

$$\frac{\partial H}{\partial x_i}(p,x) = \sum_{k=1}^{n}\left(p_k \frac{\partial Q^k}{\partial x_i}(p,x) - \frac{\partial L}{\partial q_k}(Q(p,x),x)\frac{\partial Q^k}{\partial x_i}\right) - \frac{\partial L}{\partial x_i}(Q(p,x),x)$$

(by defn of ~~p~~ p) $\longrightarrow$ $= -\frac{\partial L}{\partial x_i}(Q(p,x),x)$

Along the trajectory $(p(s), x(s))$, $Q(p(s),x(s)) = q(s) = \dot{x}(s)$, so

$$\frac{\partial H}{\partial x_i}(p(s),x(s)) = -\frac{\partial L}{\partial x_i}(Q(p(s),x(s)),x(s)) = -\frac{\partial L}{\partial x_i}(\dot{x}(s),x(s))$$

$$= -\frac{d}{ds}\frac{\partial L}{\partial q_i}(\dot{x}(s),x(s)) = -\frac{d}{ds}p(s),$$

which is the ~~fi~~ second Hamilton^ian^ equation.

For all $(p,x) \in \mathbb{R}^n \times \mathbb{R}^n$ and $\forall\ i = 1 \dots n$,

$$\frac{\partial H}{\partial p_i}(p,x) = Q^i(p,x) + \sum_{k=1}^{n}\left(p_k \frac{\partial Q}{\partial p_i}(p,x) - \frac{\partial L}{\partial q_k}(Q(p,x),x)\frac{\partial Q^k}{\partial p_i}(p,x)\right)$$

$$= Q^i(p,x), \text{ again by definition of } p.$$

Along $(p(s), x(s))$, $\frac{\partial H}{\partial p_i}(p(s),x(s)) = Q^i(p(s),x(s)) = \dot{x}(s)$.

~~$\frac{d}{ds}H(p(s),x(s))$~~ which is the first Hamilton ~~ian~~ equation.

Finally, $\frac{d}{ds}H(p(s),x(s)) = D_pH(p(s),x(s))\frac{\partial p}{\partial s}(s) + D_xH(p(s),x(s))\frac{dx}{ds}(s)$

$= 0$ because of Hamilton's equations ∎

Noether's Theorem - - -

Let $L(q,x)$ be a Lagrangian, and let G be a Lie group. Let an action $x \mapsto gx$ $(g \in G)$ be a differentiable group action by G on $\mathbb{R}^n$. Suppose that $\forall g \in G$, and for all C^1 ~~trajectory~~ curves $x(t)$,

$$L((gx)^{\cdot}(t),\ gx(t)) = L(\dot{x}(t),\ x(t)).$$

We say that the Lagrangian (or the system determined by the Lagrangian) has this G-action as a differentiable group of symmetries.

Let $g \mapsto (\beta_1 \dots \beta_m)$ be a coordinate chart for an open neighborhood of the identity $e \in G$. Set

$$gx(t) = x(t,\beta) = x(t, \beta_1 \dots \beta_m).$$

Then calculus produces conserved quantities of the system as follows. Let $x(t)$ be a soln. of the E-L equations. $\forall\, i = 1 \dots m$,

$p(t) = p(t,\beta) = D_q L(\dot{x}(t,\beta), x(t,\beta))$

$$0 = \frac{\partial}{\partial \beta_i} L(\dot{x}(t,\beta), x(t,\beta)) = D_q L(\dot{x},x) \frac{\partial}{\partial \beta_i}\frac{\partial}{\partial t} x(t,\beta) + D_x L(\dot{x},x) \frac{\partial}{\partial \beta_i}(x(t,\beta))$$

$$= p(t) \frac{\partial}{\partial t}\frac{\partial}{\partial \beta_i} x(t,\beta) + D_x L(\dot{x},x) \frac{\partial}{\partial \beta_i} x(t,\beta)$$

$$= \frac{\partial}{\partial t}\left(p(t) \frac{\partial}{\partial \beta_i} x(t,\beta)\right) - \left(\frac{dp}{dt}(t)\right) \frac{\partial}{\partial \beta_i} x(t,\beta) + D_x L(\dot{x},x) \frac{\partial}{\partial \beta_i} x(t,\beta)$$

$$= \frac{d}{dt}\left(p(t) \cdot \frac{\partial}{\partial \beta_i}(t,\beta)\right)$$ by the E-L equations.

Thus $p(t) \cdot \frac{\partial}{\partial \beta_i} x(t,\beta) \equiv \text{const}$

This is essentially Noether's Theorem.