Continuation of $\begin{cases} \dot{x} = \mathbb{A}x \\ x(0) = y \end{cases} \longrightarrow \quad x(t) = e^{t\mathbb{A}}y$

Let Λ be a normal-form matrix for the operator $\mathbb{A}$. This means that there are n independent vectors $x^{(1)}, \ldots, x^{(n)}$ such that

$$\mathbb{A}\left[x^{(1)}, \ldots, x^{(n)}\right] = \left[x^{(1)}, \ldots, x^{(n)}\right]\Lambda$$

This implies that, for all powers $m \geq 0$,

$$\mathbb{A}^m\left[x^{(1)}, \ldots, x^{(n)}\right] = \left[x^{(1)}, \ldots, x^{(n)}\right]\Lambda$$

and thus

$$e^{t\mathbb{A}}\left[x^{(1)}, \ldots, x^{(n)}\right] = \left[x^{(1)}, \ldots, x^{(n)}\right]e^{t\Lambda}$$

The initial condition y can be written as a combo of the $x^{(i)}$:

$$y = \left[x^{(1)}, \ldots, x^{(n)}\right]c = c_1x^{(1)} + \ldots + c_nx^{(n)} = Mc.$$

The solution is ....

$$x(t) = e^{t\mathbb{A}}y = e^{t\mathbb{A}}Mc = c_1e^{t\mathbb{A}}x^{(1)} + \ldots + c_ne^{t\mathbb{A}}x^{(n)}$$

The special solutions $e^{t\mathbb{A}}x^{(i)}$ are called the <u>modes</u>, or <u>normal modes</u> of the system.

The diagonal case: $\Lambda = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}$

$$e^{tA} x^{(j)} = e^{t\lambda_j} x^{(j)}$$

General soln: $x(t) = c_1 x^{(1)} e^{\lambda_1 t} + \dots + c_n x^{(n)} e^{\lambda_n t}$

A Jordan block: $\Lambda = \begin{bmatrix} \lambda & 1 & & 0 \\ & \ddots & \ddots & \\ & & & 1 \\ 0 & & & \lambda \end{bmatrix}$, $e^{t\Lambda} = e^{\lambda t} \begin{bmatrix} 1 & t & t^2/2 & \cdots & \frac{t^{n-1}}{(n-1)!} \\ & 1 & & & \vdots \\ & & \ddots & & \\ & & & 1 & t \\ & & & & 1 \end{bmatrix}$

Modes:
$$\begin{aligned} e^{tA} x^{(1)} &= e^{\lambda t} x^{(1)} \\ e^{tA} x^{(2)} &= e^{\lambda t}\left(x^{(2)} + t x^{(1)}\right) \\ e^{tA} x^{(3)} &= e^{\lambda t}\left(x^{(3)} + t x^{(2)} + \frac{t^2}{2} x^{(1)}\right) \\ &\vdots \\ e^{tA} x^{(n)} &= e^{\lambda t}\left(x^{(n)} + t x^{(n-1)} + \dots + \frac{t^{(n-1)}}{(n-1)!} x^{(1)}\right) \end{aligned}$$

Notice that the instantaneous coupling between modes given by Λ is between adjacent modes only. This has the effect of each vector $x^{(j)}$ exciting the previous ones polynomially.

Real ODEs

An ODE system in a typical application is a real system, but often we pass through a complex one in order to solve it effectively.

Let $\begin{cases} \dot{x} = Ax \\ x(0) = x_0 \end{cases}$ be an ODE system in $\mathbb{R}^n$.

The complexification of this system has exactly the same form, except now x is allowed to be in the complexified space

$$\mathbb{C}^n \equiv \mathbb{R}^n + i\mathbb{R}^n = \{ x + iy : x, y \in \mathbb{R}^n \} .$$

In $x+iy$, the vectors x and y are the real and imaginary parts.

Obs 1 If $z(t) = x(t) + iy(t)$ is a solution (with $x(t)$ and $y(t)$ in $\mathbb{R}^n$) of the complexified system $\dot{z} = Az$, then $x(t)$ and $y(t)$ are solutions of the real system, i.e., $\dot{x} = Ax$ and $\dot{y} = Ay$.

generalized e-vectors ‖ Jordan chain vectors

Obs. 2 The eigenvalues of A consist of real numbers and complex-conjugate pairs. If λ is a real eigenvalue, then the corresp. generalized eigenvectors can be taken to be real. More generally, if λ and $\bar{\lambda}$ are eigenvalues of A, then generalized eigenvectors can be chosen such that they are complex conjugates of each other.

With this in mind, consider a typical normal mode for a nonreal eigenvalue $\lambda = \alpha + i\beta$:

$$z(t) = x(t) + iy(t) = e^{\alpha t} e^{i\beta t} \left(z^{(\ell)} + t z^{(\ell-1)} + \dots + \frac{t^{(\ell-1)}}{(\ell-1)!} z^{(1)} \right)$$

Assuming the convention of Obs. 2, we have that

$$\bar{z}(t) = x(t) - i\,y(t) = e^{\alpha t} e^{-i\beta t}\left(\bar{z}^{(l)} + t\,\bar{z}^{(l-1)} + \dots + \frac{t^{(l-1)}}{(l-1)!}\,\bar{z}^{(1)}\right)$$

is a normal mode for eigenvalue $\lambda = \alpha - i\beta$.

Obs 3 $\quad x(t) = \frac{1}{2}(z(t) + \bar{z}(t))$ and $y(t) = \frac{1}{2i}(z(t) - \bar{z}(t))$ (the real and imaginary parts of $z(t)$ or $\bar{z}(t)$) form a real basis for the subspace of solutions spanned by the complex basis $z(t)$ and $\bar{z}(t)$.

It is typical in math, physics, engineering, etc., to compute solutions to the complexified system and then take the real part as the desired solution, since working with complex exponentials is simple.

For the general mode $z(t)$ above, we have

$$\begin{aligned} \operatorname{Re} z(t) = x(t) = e^{\alpha t}\Big[&\Big(x^{(l)} + t\,x^{(l-1)} + \dots + \frac{t^{(l-1)}}{(l-1)!}\,x^{(1)}\Big)\cos\beta t \;+ \\ &- \Big(y^{(l)} + t\,y^{(l-1)} + \dots + \frac{t^{(l-1)}}{(l-1)!}\,y^{(1)}\Big)\sin\beta t\Big], \end{aligned}$$

in which $x^{(j)} = \operatorname{Re} z^{(j)}$ and $y^{(j)} = \operatorname{Im} z^{(j)}$. And

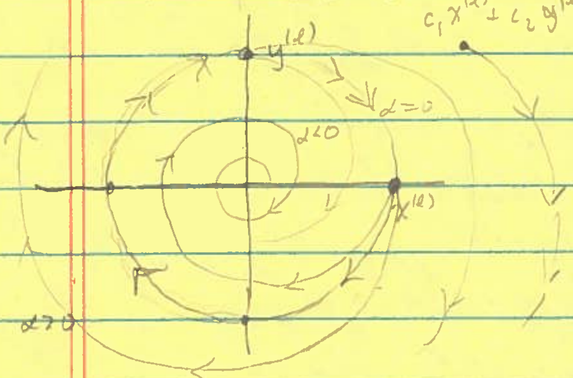
$$\begin{aligned} \operatorname{Im} z(t) = y(t) = e^{\alpha t}\Big[&\Big(y^{(l)} + t\,y^{(l-1)} + \dots + \frac{t^{(l-1)}}{(l-1)!}\,y^{(1)}\Big)\cos\beta t \;+ \\ &+ \Big(x^{(l)} + t\,x^{(l-1)} + \dots + \frac{t^{(l-1)}}{(l-1)!}\,x^{(1)}\Big)\sin\beta t\Big] \end{aligned}$$

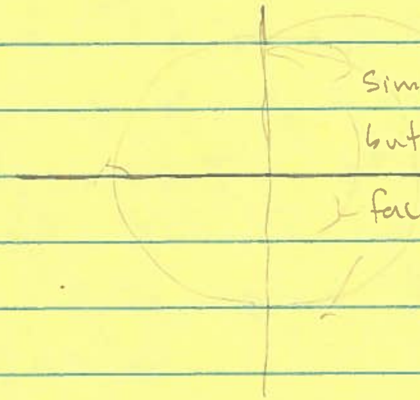
The initial conditions are just the re & imag. parts of the cx e.vec $z^{(l)}$:

$$x(0) = x^{(l)}, \qquad y(0) = y^{(l)}.$$

Graphical depiction of the solutions $x(t)$ and $y(t)$.

span$\{x^{(l)}, y^{(l)}\}$. span$\{x^{(l-1)}, y^{(l-1)}\}$





Similar picture but with a time factor t. ...

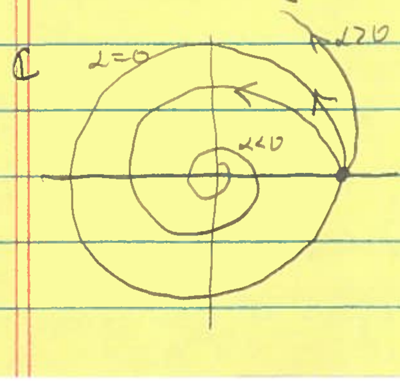
In the simpler case of a *real* eigenvalue λ, a normal mode is

$$x(t) = e^{\lambda t}\left(x^{(l)} + t\,x^{(l-1)} + \dots + \frac{t^{(l-1)}}{(l-1)!}\,x^{(1)}\right)$$

When $\lambda = 0$, the solution is polynomial in t.

Contrast this with a *purely complex* system.

1D case $\dot{z} = \lambda z$ — The operator A is just mult. in $\mathbb{C}$ by a complex number $\lambda = \alpha + i\beta$



trajectories for $\beta > 0$ and different signs of α