Consider the forced spectral problem on ~~the~~ interval $[0,1]$;

(*) $$-u'' + (q(x) - \lambda)u = f \quad , \qquad x \in [0,1]$$

Define solutions $c(x;\lambda)$ and $s(x;\lambda)$ by their initial conditions

$$c(0;\lambda) = 1 \qquad\qquad s(0;\lambda) = 0$$
$$c'(0;\lambda) = 0 \qquad\qquad s'(0;\lambda) = 1 \quad ,$$

where the prime denotes differentiation with respect to x; and define solutions $\tilde{c}(x;\lambda)$ and $\tilde{s}(x;\lambda)$ by their final conditions

$$\tilde{c}(1;\lambda) = 1 \qquad\qquad \tilde{s}(1;\lambda) = 0$$
$$\tilde{c}'(1;\lambda) = 0 \qquad\qquad -\tilde{s}'(1;\lambda) = 1$$

The values $s(1;\lambda)$ and $\tilde{s}(0;\lambda)$ are equal to $-W(s,\tilde{s})$:

$$s(1,\lambda) = -\begin{vmatrix} s(1;\lambda) & 0 \\ s'(1;\lambda) & -1 \end{vmatrix} = -W(s,\tilde{s}) = -\begin{vmatrix} 0 & \tilde{s}(0;\lambda) \\ 1 & \tilde{s}'(0,\lambda) \end{vmatrix} = \tilde{s}(0,\lambda).$$

Similarly, one can prove $c'(1;\lambda) = -\tilde{c}'(0;\lambda)$ and other relations.

The Dirichlet Green function for (*) is

$$G(x,y;\lambda) = \frac{-1}{s(1,\lambda)} \begin{cases} s(x;\lambda)\,\tilde{s}(y;\lambda), & x \le y \\ s(y;\lambda)\,\tilde{s}(x;\lambda), & y \le x \end{cases} \left(= \frac{1}{s(1,\lambda)} \tilde{G}(x,y;\lambda) \right)$$

The solution to the <u>boundary-value problem</u> (*) with $u(0) = u_0$ and $u(1) = u_1$ is

$$u(x) = \frac{1}{s(1,\lambda)} \left[u_0 \tilde{s}(x,\lambda) + u_1 s(x,\lambda) + \int_0^1 \tilde{G}(x,y;\lambda) f(y)\, dy \right]$$

The third piece of this formula satisfies the homogeneous Dirichlet boundary values — it vanishes at $x=0$ and at $x=1$ since $\tilde{G}(\cdot, y; \lambda)$ does for each $y \in (0,1)$.

This <u>Dirichlet boundary-value problem</u>, ie (*) with $u(0) = u_0$ and $u(1) = u_1$ is uniquely solvable whenever $s(1,\lambda) \neq 0$. Those values of λ for which $s(1,\lambda) = 0$ are called the Dirichlet eigenvalues of $-\frac{d^2}{dx^2} + q(x)$ on $[0,1]$; the eigenfunctions are $s(x;\lambda)$.

$$s(1,\lambda) = 0 \iff \lambda \in \sigma_D\left(-\frac{d^2}{dx^2} + q\right)$$

where σ_D denotes the "Dirichlet spectrum".

<u>Case $q(x) = 0$</u>

In this case, $c(x;\lambda) = \cos\sqrt{\lambda}\, x$, $s(x;\lambda) = \frac{\sin\sqrt{\lambda}\, x}{\sqrt{\lambda}}$

$\tilde{c}(x;\lambda) = \cos\sqrt{\lambda}(1-x)$, $\tilde{s}(x;\lambda) = \frac{1}{\sqrt{\lambda}} \sin\sqrt{\lambda}(1-x)$

and the Dirichlet spectral function is $s(1,\lambda) = \frac{\sin\sqrt{\lambda}}{\sqrt{\lambda}} =: \operatorname{sinc}\sqrt{\lambda}$

Notice that for each x, each of these solutions is an <u>analytic function</u> of λ [prove this directly] on $\mathbb{C}$.

In fact, for each x, the functions $c, s, \tilde{c}, \tilde{s}$ are entire functions of λ ~ this is the content of one of your assignments.

The Dirichlet Green function for $q(x) \equiv 0$ is

$$G_0(x,y;\lambda) = \frac{\sqrt{\lambda}}{\sin\sqrt{\lambda}} \begin{cases} \dfrac{\sin\sqrt{\lambda}\,x}{\sqrt{\lambda}} \sin\dfrac{\sqrt{\lambda}(1-y)}{\sqrt{\lambda}}, & x \le y \\[2ex] \dfrac{\sin\sqrt{\lambda}\,y}{\sqrt{\lambda}} \sin\dfrac{\sqrt{\lambda}(1-x)}{\sqrt{\lambda}}, & x > y \end{cases}$$

$$= \frac{1}{\sqrt{\lambda}\,\sin\sqrt{\lambda}} \begin{cases} \sin\sqrt{\lambda}\,x \,\sin\sqrt{\lambda}(1-y), & x \le y \\ \sin\sqrt{\lambda}\,y\,\sin\sqrt{\lambda}(1-x), & x \ge y \end{cases}$$

Notice that $G_0(\cdot,\cdot;\lambda)$ has poles, as a function of λ, at the Dirichlet spectrum of $-\frac{d^2}{dx^2}$ on $[0,1]$, namely

$$\sigma_D\left(-\frac{d^2}{dx^2}\right) = \left\{\pi^2, (2\pi)^2, (3\pi)^2, \dots\right\}$$

Recall the <u>causal fundamental solution</u>

$$\Phi(x,y) = -\frac{1}{\sqrt{\lambda}} \sin\sqrt{\lambda}(x-y)\,\chi_{\{y \le x\}}$$

and the ~~corresponding~~ resulting solution to the initial-value problem

$$-u'' - \lambda u = 0, \quad u(0) = u_0, \quad u'(0) = u_0',$$

which is

$$u(x) = u_0 \cos\sqrt{\lambda}x + u_0' \frac{\sin\sqrt{\lambda}x}{\sqrt{\lambda}} - \int_0^x \frac{\sin\sqrt{\lambda}(x-y)}{\sqrt{\lambda}} f(y)\,dy.$$

The "anti-causal" fundamental solution is

$$\tilde{\Phi}(x,y;\lambda) = \frac{1}{\sqrt{\lambda}} \sin\sqrt{\lambda}(x-y)\, \chi_{\{x<y\}},$$

and the solution to the final-value problem

$$-u'' - \lambda u = 0, \quad u(1) = u_1, \quad u'(1) = u_1'$$

is

$$u(x) = u_1 \cos\sqrt{\lambda}(1-x) - u_1' \frac{\sin\sqrt{\lambda}(1-x)}{\sqrt{\lambda}} + \int_x^1 \frac{\sin\sqrt{\lambda}(x-y)}{\sqrt{\lambda}} f(y)\,dy$$

Let's apply these formulas to the initial-value problems with general potential $q(x)$ but with no forcing f,

$$-u'' + (q(x) - \lambda)u = 0,$$

which is equivalently

$$-u'' \cancel{+} - \lambda u = -q(x)u$$

Considering the initial (final) values for $c, s, \tilde{c}, \hat{s}$, we obtain

$$c(x;\lambda) = \cos\sqrt{\lambda}\,x + \int_0^x \frac{\sin\sqrt{\lambda}\,(x-y)}{\sqrt{\lambda}}\, q(y)\, c(y;\lambda)\, dy$$

$$s(x;\lambda) = \frac{\sin\sqrt{\lambda}\,x}{\sqrt{\lambda}} + \int_0^x \frac{\sin\sqrt{\lambda}\,(x-y)}{\sqrt{\lambda}}\, q(y)\, s(y;\lambda)\, dy$$

$$\tilde{c}(x;\lambda) = \cos\sqrt{\lambda}\,(1-x) - \int_x^1 \frac{\sin\sqrt{\lambda}\,(x-y)}{\sqrt{\lambda}}\, q(y)\, \tilde{c}(y;\lambda)\, dy$$

$$\hat{s}(x;\lambda) = \frac{\sin\sqrt{\lambda}\,(1-x)}{\sqrt{\lambda}} - \int_x^1 \frac{\sin\sqrt{\lambda}\,(x-y)}{\sqrt{\lambda}}\, q(y)\, \tilde{s}(y;\lambda)\, dy .$$

All of these are ~~integ~~ Volterra integral equations for these special solutions. Let's look at the first one.

By setting $\dfrac{\sin\sqrt{\lambda}\,(x-y)}{\sqrt{\lambda}}\, q(y) =: Q(x,y;\lambda)$, one has

$$c(x;\lambda) - \int_0^x Q(x,y;\lambda)\, c(y;\lambda)\, dy = \cos\sqrt{\lambda}\,x ,$$

or, more compactly,

$$[(E - Q(\lambda))\,c](x;\lambda) = \cos\sqrt{\lambda}\,x$$

The kernel $Q(x,y;\lambda)$, as a function of x and y is in $L^2([0,1]^2)$, and therefore one may apply the L^2 theory of Volterra integral operators.

The equation $(\mathbb{E} - \mathbb{Q}(\lambda))\, c(\cdot;\lambda) = \cos\sqrt{\lambda}\,\cdot$
has a unique solution $c(\cdot;\lambda)$ in L^2, which is in fact absolutely continuous:

$$c(\cdot;\lambda) = [\mathbb{E} - \mathbb{Q}(\lambda)]^{-1} \cos\sqrt{\lambda}\,\cdot$$

The inverse $[\mathbb{E} - \mathbb{Q}(\lambda)]^{-1}$ is also $\mathbb{E}$ plus a Volterra integral operator:

$$[\mathbb{E} - \mathbb{Q}(\lambda)]^{-1} = \mathbb{E} + \sum_{n=1}^{\infty} \mathbb{Q}(\lambda)^n = \mathbb{E} + \mathbb{K}(\lambda) .$$

We have already shown how to compute the kernel of $\mathbb{K}(\lambda)$ by summing the kernels of $\mathbb{Q}(\lambda)^n$, called $Q_n(x,y;\lambda)$: Let the integral kernel for $\mathbb{K}$ be called $K(x,y;\lambda)$. Then

$$K(x,y;\lambda) = \sum_{n=1}^{\infty} Q_n(x,y;\lambda)$$

The REMARKABLE THEOREM is that in fact <u>$K(x,y;\lambda)$ does not depend on λ</u>: $K(x,y;\lambda) = K(x,y)$.

<u>Theorem</u> If $q \in L^2[0,1]$, then $K(x,y) \in L^2([0,1]^2)$ and

$$c(x;\lambda) = \cos\sqrt{\lambda}\,x + \int_0^x K(x,y) \cos\sqrt{\lambda}\,y \, dy$$

Also, $K(x,x) = \frac{1}{2}\int_0^x q(y)\,dy$.

By a similar argument, there is an L^2 kernel $L(x,y)$ such that

$$s(x;\lambda) = \frac{\sin\sqrt{\lambda}\,x}{\sqrt{\lambda}} + \int_0^x L(x,y)\,\frac{\sin\sqrt{\lambda}\,y}{\sqrt{\lambda}}\,dy$$

Again, $L(x,y)$ is independent of λ.

From this, one obtains the important asymptotic formula

$\rho^2 = \lambda$

$$\begin{aligned} s(x;\lambda) &= \frac{\sin\rho x}{\rho} + O\left(\frac{1}{|\rho|^2}\exp(|\operatorname{Im}\rho|x)\right) \\ &= \frac{1}{\rho}\left[\sin\rho x + O\left(\frac{1}{|\rho|}\exp(|\operatorname{Im}\rho|x)\right)\right] \\ &= O\left(\frac{1}{|\rho|}\exp(|\operatorname{Im}\rho|x)\right) \end{aligned}$$

An analogous result holds for $\tilde{s}(x;\lambda)$ (essentially replace x with $1-x$).

Let us return to the fixed spectral problem

$$-u'' + (q(x)-\lambda)u = f, \quad u_0 = 0,\ u_1 = 0$$

Using the Green function at the bottom of p. 1, we obtain the solution when $s(1,\lambda) \neq 0$:

$$(u) \qquad u(x;\lambda) = \frac{-1}{s(1;\lambda)}\left[s(x;\lambda)\int_x^1 \tilde{s}(y;\lambda)f(y)\,dy + \tilde{s}(x;\lambda)\int_0^x s(y;\lambda)f(y)\,dy\right]$$

$$= \frac{-1}{s(1;\lambda)}\,\tilde{u}(x;\lambda)$$

In the theory of entire functions, Hadamard's Factorization Theorem says that if an entire function $f : \mathbb{C} \to \mathbb{C}$ has growth order γ, meaning that

$$|f(z)| \le Ae^{B|z|^\gamma}$$

for some constants A and B, then f has a representation as an infinite product

$$f(z) = e^{P(z)} z^m \prod_{n=1}^{\infty} E_k(z/a_n),$$

in which k is the integer such that $k \le \gamma < k+1$, $a_1, a_2 \ldots$ are the nonzero roots of f, P is a polynomial of ~~order~~ degree $\le k$, and m is the order of the root of f at $z=0$, and the "canonical factors" $E_k(z)$ are defined by

$$E_0(z) = 1 - z, \quad E_k(z) = (1-z)e^{z + \frac{z^2}{2} + \cdots + \frac{z^k}{k}} \quad (k \ge 1).$$

The asymptotic form on p. 7 for $s(x;\lambda)$ shows that, for fixed x, $s(x;\lambda)$ is of growth order $1/2$. From this, one can conclude that $s(1;\lambda)$ has infinitely many roots. Moreover the roots are real, as we have proved. They are also <u>positive</u> (prove this as an exercise).

Let the roots of $s(x;\lambda)$, which are the Dirichlet eigenvalues of $-d^2/dx^2 + q(x)$ on $[0,1]$, be denoted by

$$\lambda_n\,, \quad n=1,2,3,\ldots\,, \qquad s(1;\lambda_n)=0\,.$$

In addition, one can prove that

$$\sum_{n=1}^{\infty} \frac{1}{\lambda_n^{1/2+\varepsilon}} < \infty$$

for all $\varepsilon>0$. See, for example, Chapter 5 of Complex Analysis by Stein & Shakarchi.

Going back to (u), p. 7, and using the asymptotic form of $s(x;\lambda)$ and $\tilde{s}(x;\lambda)$ on p 7, one can deduce ~~that~~

(p) Proposition There is a constant $C>0$ and a constant $R>0$ such that, for all ~~$\lambda\in\mathbb{C}$ such that $|\rho|>R$~~ $\rho^2=\lambda\in\mathbb{C}$, with $\rho>R$ and $|\rho-k\pi|>1/2 \;\; \forall\, k\in\mathbb{Z}$,

$$|u(x;\lambda)| \le \frac{C}{|\rho|} \qquad \rho^2=\lambda\,.$$

The residue of $u(x;\lambda)$ at the root λ_n of $s(1,\lambda)$ is

$$\operatorname*{Res}_{\lambda=\lambda_n} u(x;\lambda) = \frac{-1}{\frac{d}{d\lambda}s(1,\lambda_n)}\,\tilde{u}(x;\lambda_n)\,,$$

in which $\tilde{u}(x;\lambda)$ is the regular factor of u at the bottom of p. 7. So

$$\tilde{u}(x;\lambda_n) = \beta_n s(x;\lambda_n)\int_0^1 s(y;\lambda_n) f(y)\,dy$$

<u>Exercise</u> Prove that

$$\frac{d}{d\lambda} s(1,\lambda_n) = -\beta_n \alpha_n ,$$

where β_n and α_n are defined by

$$\tilde{s}(x;\lambda_n) = \beta_n s(x;\lambda_n) \quad \text{and} \quad \alpha_n = \int_0^1 s(x;\lambda_n)^2\,dx$$

Putting these results together gives

* $$\operatorname*{Res}_{\lambda=\lambda_n} u(x,\lambda) = \frac{1}{\alpha_n} s(x;\lambda_n)\int_0^1 s(y;\lambda_n) f(y)\,dy$$

<u>Theorem</u> The eigenfunctions $\{ s(x;\lambda_n) : n = 1, 2, 3, \dots \}$ are complete in $L^2([0,1])$ in the sense that if $\int_0^1 s(x;\lambda_n) f(x)\,dx = 0$ for all n, then $f = 0$ in $L^2([0,1])$.

<u>Proof</u> Let $f \in L^2([0,1])$ be such that $\int_0^1 s(x;\lambda_n) f(x)\,dx = 0$ for all n. Then $\operatorname*{Res}_{\lambda=\lambda_n} u(x;\lambda) = 0$ at all poles λ_n of $u(x;\lambda)$.

This implies that $u(x;\lambda)$ is an entire function. In view of ~~the limit~~ proposition (p2), p. 9, and the maximum principle (an analytic function in a bounded domain attains its maximal value on the boundary of the domain), we obtain $u(x;\lambda) \to 0$ as $|\lambda| \to \infty$. Since $u(x;\lambda)$ is a bounded entire function of λ (with x fixed), Liouville's Theorem implies that $u(x;\lambda)$ is independent of λ. And since $u(x;\lambda) \to 0$, in fact $u(x;\lambda) \equiv 0$. By the ODE for u, we find that $f(x) \equiv 0$. ∎

<u>Theorem</u> For all $f \in L^2([0,1])$,

$$f = \sum_{n=1}^{\infty} \frac{s(x;\lambda_n)}{\sqrt{\alpha_n}} \int_0^1 \frac{s(y;\lambda_n)}{\sqrt{\alpha_n}} f(y)\,dy$$

in $L^2([0,1])$ (ie the convergence is in the L^2 norm).

<u>Sketch of proof</u>

<u>Step 1</u> Using the representation of the response $u(x;\lambda)$ to $f(x)$, ie the solution of $-u'' + (q(x)-\lambda)u = f$, one obtains by integration by parts, in the case that f is absolutely continuous,

$$\begin{aligned}
-\Delta(\lambda)\,u(x;\lambda) &= s(x;\lambda)\int_x^1 \tilde{s}(y,\lambda) f(y)\,dy + \tilde{s}(x,\lambda)\int_0^x s(y,\lambda) f(y)\,dy \\
&= \frac{1}{\lambda}\Big[s(x,\lambda)\int_x^1 (-\tilde{s}'' + q\tilde{s}) f\,dy + \tilde{s}(x,\lambda)\int_0^x (-s'' + qs) f\,dy \\
&= \frac{1}{\lambda}\Big[s(x,\lambda)\int_x^1 q(y)\tilde{s}(y,\lambda) f(y)\,dy + \tilde{s}(x,\lambda)\int_0^x q(y) s(y,\lambda) f(y)\,dy\Big] + \\
&\quad + \frac{1}{\lambda}\Big[s(x,\lambda)\int_x^1 \tilde{s}'(y,\lambda) f'(y)\,dy + \tilde{s}(x,\lambda)\int_0^x s'(y,\lambda) f'(y)\,dy\Big] + \\
&\quad + \frac{1}{\lambda}\Big[s(x,\lambda)\big(f(0) + f(1)\big) + \big(s(x)\tilde{s}'(1,\lambda) - \tilde{s}(x,\lambda)\, s'(x,\lambda)\big) f(1)\Big)
\end{aligned}$$

$$= -\frac{1}{\lambda}\Delta(\lambda) f(x) + \frac{1}{\lambda}\left(s(x)\,(f(0)+f(1))\right) +$$

$$+ \frac{1}{\lambda}\int_0^1 q(y)\,\tilde{G}(x,y)\,f(y)\,dy +$$

$$+ \frac{1}{\lambda}\left[s(x,\lambda)\int_x^1 \tilde{s}'(y,\lambda) f'(y)\,dy + \tilde{s}(x,\lambda)\int_0^x s'(y,\lambda) f'(y)\,dy\right]$$

Thus

$$u(x,\lambda) = \frac{1}{\lambda} f(x) - \frac{1}{\lambda}\frac{1}{\Delta(\lambda)}\Big[\int_0^1 q(y)\,\tilde{G}(x,y)\,f(y)\,dy + s(x)\,(f(0)+f(1)) +$$

$$+ s(x)\int_x^1 \tilde{s}'(y)\,f'(y)\,dy + \tilde{s}(x)\int_0^x s'(y)\,f'(y)\,dy\Big]$$

Given that f is absolutely continuous, one shows that

(*) $$\frac{1}{\Delta(\lambda)}[\ldots] \to 0 \quad \text{as} \quad |\rho| \to \infty \qquad (\rho^2 = \lambda)$$
for $|\rho - n\pi| < \frac{1}{2}\ \forall n$.

where $[\ldots]$ is the stuff in brackets in the expression for u. Showing (*) takes some technical work (see Freiling/Yurko p. 16 ...).

This convergence in (*) is uniform over $x \in [0,1]$.

Step 2 Consider the integrals

$$I_n(x) = \frac{1}{2\pi i}\oint_{\{\lambda = (n\pi + \frac{1}{2})^2\}} u(x,\lambda)\,d\lambda$$

From (*), one finds that

$$I_n(x) = f(x) + \varepsilon_n(x) \quad ,$$

in which $\varepsilon_n(x) \to 0$ as $n \to \infty$ uniformly over $x \in [0,1]$.
Using the residue theorem, one obtains

$$I_n(x) = \sum_{k=1}^{n} \frac{s(x,\lambda_k)}{\sqrt{\alpha_k}} \int_0^1 \frac{s(y,\lambda_k)}{\sqrt{\alpha_k}} f(y)\,dy \quad .$$

By taking $n \to \infty$, one obtains

(†) $$f(x) = \sum_{n=1}^{\infty} \frac{s(x,\lambda_n)}{\sqrt{\alpha_n}} \int_0^1 \frac{s(y,\lambda_n)}{\sqrt{\alpha_n}} f(y)\,dy \quad ,$$

in which the convergence of the sum is uniform in $[0,1]$.

The uniform convergence of (†) (for f absolutely continuous) implies L^2 convergence.

Step 3 Since (†) converges in L^2 for a dense subset (abs. cont. fns) of $L^2[0,1]$, (†) is actually valid for all $f \in L^2$. (You can fill in the details.)

Theorem (Unique determination of q from spectral data)
Let q_1 and $q_2 \in L^2([0,1])$ be given, and let $\{\lambda_k, \alpha_k\}_{k=1}^{\infty}$ be equal to the Dirichlet spectrum and square norms of the eigenfunctions for both $-\frac{d^2}{dx^2} + q_1(x)$ and $-\frac{d^2}{dx^2} + q_2(x)$ on $[0,1]$.
Then $q_1 \equiv q_2$ in $L^2[0,1)$.

Sketch of proof Let $s_1(x,\lambda)$ and $s_2(x,\lambda)$ denote the s-functions for q_1 and q_2 respectively. We have seen that there are Volterra integral operators ~~T_1 call T_2 with~~ $\mathbb{K}_1$ and $\mathbb{K}_2$ with L^∞ kernels such that $\forall \lambda$,

$$s_1(\cdot,\lambda) = (\mathbb{E} + \mathbb{K}_1)\frac{\sin(\sqrt{\lambda}\,\cdot)}{\sqrt{\lambda}}$$

$$s_2(\cdot,\lambda) = (\mathbb{E} + \mathbb{K}_2)\frac{\sin(\sqrt{\lambda}\,\cdot)}{\sqrt{\lambda}}.$$

Thus $$s_2(\cdot,\lambda) = (\mathbb{E} + \mathbb{T})\, s_1(\cdot,\lambda)$$

for all λ, where $\mathbb{T}$ is a Volterra integral operator with L^2 kernel that is independent of λ; call it $T(x,y)$.

Let $(\cdot,\cdot)$ denote the inner product in $L^2([0,1])$.
We have for all $f \in L^2$,

$$(s_2(\cdot,\lambda), f) = ((\mathbb{E}+\mathbb{T})s_1(\cdot,\lambda), f)$$

$$= (s_1(\cdot,\lambda), (\mathbb{E}+\mathbb{T}^*)f) = (s_1(\cdot,\lambda), g),$$

in which $\mathbb{T}^*$ is defined by

$$(\mathbb{T}^* f)(x) = \int_x^1 T(y,x)\, f(y)\, dy \; .$$

[You can check that this is valid. { $\mathbb{T}^*$ is called the adjoint of $\mathbb{T}$ }] and $g = (\mathbb{E} + \mathbb{T}^*) f$.

In particular, we obtain

$$a_n := \left(\frac{s_2(\cdot, \lambda_n)}{\sqrt{\alpha_n}}, f \right) = \left(\frac{s_1(\cdot, \lambda_n)}{\sqrt{\alpha_n}}, g \right) =: b_n$$

for all $n = 1, 2, 3, \ldots$. Thus,

$$(f, f) = \sum_{n=1}^{\infty} |a_n|^2 = \sum_{n=1}^{\infty} |b_n|^2 = (g, g)$$

(using Parseval's identity coming from the previous theorem)

Since this holds for all $f \in L^2(0,1)$, $\mathbb{E} + \mathbb{T}^*$ is a unitary operator. It follows that $\mathbb{T}^* = 0$, so that $T(x,y) \equiv 0$. Thus $s_2(x,\lambda) = s_1(x,\lambda)$ for all λ. This can only happen when $q_1 = q_2$ in L^2.

There is a special situation in which the spectrum of a Schrödinger operator determines the potential. For the Neumann eigenvalue problem $-u'' + (q(x) - \lambda)u = 0$ with $u'(0) = u'(1) = 0$, there is only one potential $q \in L^2(0,1)$ such that the Neumann eigenvalues $\{\mu_n\}_{n=0}^{\infty}$ are equal to $(\pi n)^2$ for $n = 0, 1, 2, 3, \ldots$.

The Neumann eigenvalues are the roots of $c'(1,\lambda)$; call them $\{\mu_n\}_{n=0}^{\infty}$:

$$c'(1, \mu_n) = 0, \quad n = 0, 1, 2, \ldots$$

When $q = 0$, certainly $\mu_n = (n\pi)^2$. The converse is also true. The result is called Ambarzumian's Theorem (Thm. 1.4.1 in Freiling-Yurko)

There are two theorems that precede this one.

Thm 1.2.2 in F-Y

__Theorem__ $c(x, \mu_n)$ has exactly $n-1$ roots in $[0,1]$, and the roots of $c(x,\mu_n)$ and $c(x, \mu_{n+1})$ interlace.

The proof is not hard, and can be found in many books on Sturm-Liouville problems, including Freiling-Yurko.

Thm. 1.1.3 FY

Theorem The Neumann eigenvalues satisfy

$$\sqrt{\mu_n} = \pi n + \frac{1}{n}\int_0^1 q(x)\,dx + \frac{\kappa_n}{n},$$

in which $\kappa_n \in \ell^2(\mathbb{N})$.

The proof is rather involved and delicate, and it features a nice application of Rouché's theorem from complex variables.

Thm 1.4.1 FY

Theorem (Ambarzumian's Theorem)
If the Neumann eigenvalues of $-\frac{d^2}{dx^2} + q(x)$ on $[0,1]$ are equal to $\mu_n = (\pi n)^2$, $n = 0, 1, 2, \ldots$, then $q(x) = 0$ in $L^2[0,1]$.

Proof Assume $\mu_n = (\pi n)^2$. ~~Then the lowest eigenvalue is equal to 0, label them~~ By the previous theorem,

$$\int_0^1 q(x)\,dx = 0.$$

Since the ~~first~~ lowest eigenvalue is equal to 0, with eigenfunction $u(x) = c(x,0)$, one has

$$-u'' + q(x)u = 0, \quad u'(0) = 0 = u'(1),$$

and by the pen-previous theorem, $u(x) \neq 0 \ \forall\, x \in [0,1]$.

Using $q = \frac{u''}{u} = \left(\frac{u'}{u}\right)^2 + \left(\frac{u'}{u}\right)'$, we obtain

$$0 = \int_0^1 q(x)\,dx = \int_0^1 \left(\frac{u'(x)}{u(x)}\right)^2 dx$$

because $\int_0^1 \left(\frac{u'}{u}\right)' = \left(\frac{u'}{u}\right)\Big|_0^1 = 0$.

This implies that $u'(x) \equiv 0$ on $[0,1]$, so that $u(x) = u(0) = 1$ a.e. $x \in [0,1]$. The ODE then implies that $q(x) = 0$ on $[0,1]$. ∎

Thus, among all potentials with $\int_0^1 q(x)\,dx$ equal to a fixed constant C, the lowest Neumann eigenvalue is maximized when $\lambda_0 = C$ and $q(x) \equiv C$.

Theorem Let λ_0 be the ~~smallest~~ lowest Neumann eigenvalue of $-\frac{d^2}{dx^2} + q(x)$ on $[0,1]$. Then $\lambda_0 \le \int_0^1 q(x)\,dx$, and equality is achieved when $q(x) \equiv \lambda_0$.

Pf Let $u(x) = c(x,\lambda_0)$ satisfy $-u'' + (q(x) - \lambda_0)u = 0$ and $u(0) = 1$ and $u'(0) = 0$. As in the proof of the previous theorem (with $q(x)$ replaced by $q(x) - \lambda_0$), we obtain

$$\int_0^1 (q(x) - \lambda_0)\,dx = \int_0^1 \left(\frac{u'(x)}{u(x)}\right)^2 dx \ge 0,$$

which implies that

$$\lambda_0 \le \int_0^1 q(x)\,dx.$$

If $\int_0^1 q(x)\,dx = \lambda_0$, then $u'(x) = 0$, so $u(x) = u(1) = 1$ on $[0,1]$. Thus, from the DE, $q(x) \equiv \lambda_0$ on $[0,1]$. ∎