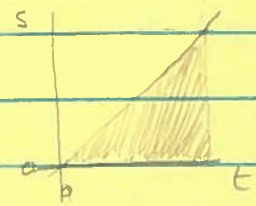
Integral operators of Volterra type

$$(\mathbb{K}\phi)(t) = \int_0^t K(t,s)\,\phi(s)\,ds, \qquad K(t,s) = 0 \text{ for } t < s.$$



$$x(t) - \int_0^t K(t,s)\,x(s)\,ds = f(t)$$

$$\Leftrightarrow \quad (\mathbb{E} - \mathbb{K})x = f$$

If inverse exists, $x = (\mathbb{E}-\mathbb{K})^{-1}f$, and should have

$$(\mathbb{E}-\mathbb{K})^{-1} = \mathbb{E} + \mathbb{K} + \mathbb{K}^2 + \dots \qquad \text{(Neumann series)}$$

The L^2 setting

Assume $K(t,s) \in L^2(I \times I)$, $I = [0,L]$ (generous assumption)

So $\displaystyle\int_{I\times I} |K(t,s)|^2\, d(t,s) < \infty$

$$\text{Fubini} \Rightarrow \quad \begin{array}{l} \displaystyle\int_I |K(t,s)|^2\,ds = A(t)^2 \\ \displaystyle\int_I |K(t,s)|^2\,dt = B(s)^2 \end{array} \quad \text{exist a.e.}$$

and $\displaystyle\int_I A(t)^2\,dt = \int_I B(t)^2\,dt = \int_{I\times I} |K(t,s)|^2\,d(t,s)$

i.e., $\|A\|_{L^2(I)} = \|B\|_{L^2(I)} = \|K\|_{L^2(I\times I)}$

$\mathbb{K}$ as an operator in $L^2(I)$:

$$|(\mathbb{K}\phi)(t)|^2 = \left|\int_I K(t,s)\phi(s)\,ds\right|^2 \le \int_I |K(t,s)|^2 ds \int_I |\phi(s)|^2 ds$$

$$\|\mathbb{K}\phi\|^2_{L^2(I)} = \int_I |\mathbb{K}\phi(t)|^2 dt \le \|K\|_2^2 \, \|\phi\|_2^2$$

$$\Longrightarrow \mathbb{K} : L^2 \to L^2 \text{ is bdd with } \|\mathbb{K}\| \le \|K\|_2$$

Iteration of L^2 operators $\mathbb{G}_1$ and $\mathbb{G}_2$ with integral kernels $G_1(t,s)$ and $G_2(t,s)$ in $I \times I$.

$$(\mathbb{G}_2\mathbb{G}_1\phi)(t) = \int_0^t G_2(t,r)\,\mathbb{G}_1\phi(r)\,dr$$

$$= \int_0^t G_2(t,r)\int_0^r G_1(r,s)\phi(s)\,ds\,dr = \int_0^t \phi(s)\int_s^t G_2(t,r)G_1(r,s)\,dr\,ds$$

$\longrightarrow$ The integral kernel for $\mathbb{G}_2\mathbb{G}_1$ is $\int_s^t G_2(t,r)G_1(r,s)\,dr \in L^2(I\times I)$

ex: Prove $\left\|\int_s^t G_2(t,r)G_1(r,s)\,dr\right\|_{L^2(I\times I)} \le \|G_2\|_{L^2}\,\|G_1\|_{L^2}$.

The integral kernels $K_n(t,s)$ for $\mathbb{K}^n$ are obtained recursively:

$$K_{n+1}(t,s) = \int_s^t K_n(t,r)K(r,s)\,dr = \int_s^t K(t,r)K_n(r,s)\,dr$$

Crude bounds:
$$\begin{array}{ccc} \|\mathbb{K}^n\| & \le & \|\mathbb{K}\|^n \\ \le & & \le \\ \|K_n\|_{L^2} & \le & \|K\|^n_{L^2} \end{array}$$

These do not take the Volterra form into account.

We will now use the Volterra property of $K(t,s)$, that is, that $K(t,s)=0$ for $t<s$, to obtain the much stronger bound

$$\|\mathbb{K}^n\| \leq \frac{\|K\|_2^n}{\sqrt{(n-2)!}}, \quad n \geq 2.$$

For $n \geq 1$: $K_{n+1}(t,s) = \int_s^t K(t,r)K_n(r,s)\,dr$

$$\Longrightarrow |K_{n+1}(t,s)|^2 \leq \int_s^t |K(t,r)|^2 dr \int_s^t |K_n(r,s)|^2 dr \leq A(t)^2 \int_s^t |K_n(r,s)|^2 dr$$

$$K_{n+1}(t,s) = \int_s^t K_n(t,r)K(r,s)\,ds \quad \leq' \quad B(s)^2 \int_s^t |K_n(t,r)|^2 dr$$

Using both of these inequalities, we obtain

$$|K_{n+2}(t,s)|^2 \leq A(t)^2 \int_s^t |K_{n+1}(r,s)|^2 dr$$

$$\leq A(t)^2 B(s)^2 \int_s^t \int_s^r |K_n(r,r')|^2 dr' dr = A(t)^2 B(s)^2 D_n(t,s)$$

where $D_n(t,s) := \int_s^t \int_s^r |K_n(r,r')|^2 dr' dr$

By defining $D_0(t,s) \equiv 1$, we obtain, for $n \geq 0$,

for $n \geq 0$ $$|K_{n+2}(t,s)|^2 \leq A(t)^2 B(s)^2 D_n(t,s) \qquad (*)$$

Analysis of $D_n(t,s)$

$$D_1(t,s) = \int_s^t \int_s^r |K(r,r')|^2 \, dr' \, dr \le \int_s^t A(r)^2 \, dr = \int_s^t A(r)^2 D_0(r,s) \, dr$$

$n \geq 1$ $$D_{n+1}(t,s) = \int_s^t \int_s^r |K_{n+1}(r,r')|^2 \, dr' \, dr$$

$$\le \int_s^t \int_s^r A(r)^2 \int_{r'}^r |K_n(r'',r')|^2 \, dr'' \, dr' \, dr$$

$$= \int_s^t A(r)^2 \int_s^r \int_s^{r''} |K_n(r'',r')|^2 \, dr' \, dr'' \, dr$$

$$= \int_s^t A(r)^2 D_n(r,s) \, dr$$

Define $\tilde{D}_0(t,s) \equiv 1$

$n \geq 0$ $$\tilde{D}_{n+1}(t,s) = \int_s^t A(r)^2 \tilde{D}_n(r,s) \, dr$$

$$\Longrightarrow D_n(t,s) \le \tilde{D}_n(t,s)$$

Integral calculus $\rightsquigarrow$ $\tilde{D}_n(t,s) = \dfrac{\tilde{D}_1^n(t,s)}{n!}$

$$\left[\begin{array}{l} F_0(t) \equiv 1 \\ F_{n+1}(t) = \int_0^t g(s) F_n(s) \, ds \\ \Longrightarrow F_n(t) = \dfrac{F_1^n(t)}{n!} \end{array} \right.$$

Back to $(*)$:

$$|K_{n+2}(t,s)|^2 \le A(t)^2 B(s)^2 \tilde{D}_n(t,s) = A(t)^2 B(s)^2 \frac{\tilde{D}_1^n(t,s)}{n!}$$

Since $\tilde{D}_1(t,s) = \int_s^t A(r)^2 \, dr \le \|K\|_2^2$, we obtain

$$|K_{n+2}(t,s)| \le A(t) B(s) \frac{\|K\|_2^n}{\sqrt{n!}} \qquad (\dagger)$$

$$(+) \qquad |K_{n+2}(t,s)| \leq A(t)\,B(s)\,\frac{\|K\|_2^n}{\sqrt{n!}} \qquad \text{for a.a. } t, s$$

By squaring (+), integrating over I and then over $I \times I$, and extracting $\sqrt{\ }$, one obt

$$(++) \qquad \|K_{n+2}(t,\cdot)\|_{L^2(I)} \leq A(t)\,\frac{\|K\|_2^{n+1}}{\sqrt{n!}} \qquad \text{for a.a. } t$$

$$(+++) \qquad \|K_{n+2}\|_{L^2(I\times I)} \leq \frac{\|K\|_2^{n+2}}{\sqrt{n!}} \quad \text{so} \quad \|\mathbb{K}^{n+2}\| \leq \frac{\|K\|^{n+2}}{\sqrt{n!}}.$$

From these bounds, we obtain absolute convergence of the kernel function $\sum_{n=0}^{\infty} K_n(t,s)$ and the associated operator $\sum_{n=0}^{\infty} \mathbb{K}^n$

(a) Pointwise $\sum_{n=0}^{\infty} K_n(t,s)$ converges absolutely, and

$$\left| \sum_{n=0}^{\infty} K_{n+2}(t,s) \right| \leq A(t)\,B(s) \sum_{n=0}^{\infty} \frac{\|K\|_2^n}{\sqrt{n!}} = K_0\, A(t)\, B(s)$$

(b) In $L^2(I)$ for each $t \in I$ $\sum_{n=0} K_n(t,\cdot)$ converges absolutely, and

$$\left\| \sum_{n=0}^{\infty} K_{n+2}(t,\cdot) \right\|_{L^2(I)} \leq \sum_{n=0}^{\infty} \|K_{n+2}(t,\cdot)\|_{L^2(I)} \leq A(t)\, K_0 \|K\|_2$$

(c) In $L^2(I\times I)$, $\sum_{n=0} K(\cdot,\cdot)$ converges absolutely, and

$$\left\| \sum_{n=0}^{\infty} K_{n+2} \right\|_{L^2(I\times I)} \leq \sum_{n=0}^{\infty} \|K_{n+2}\|_{L^2(I\times I)} \leq K_0 \|K\|_2^2$$

(d) Operators in $L^2(I)$. $\sum_{n=0}^{\infty} \mathbb{K}^n$ converges absolutely, and

$$\left\| \sum_{n=0}^{\infty} \mathbb{K}^{n+2} \right\|_{L^2 \to L^2} \leq \sum_{n=0}^{\infty} \|\mathbb{K}^{n+2}\| \leq K_0 \|K\|_2^2$$

We now show that $\sum_{n=1}^{\infty} K_n(t,s)$ is the integral kernel for $\sum_{n=1}^{\infty} \mathbb{K}^n$.

For almost all $t \in I$, item (b) and the duality between $L^2(I)$ and itself gives

$$\sum_{n=1}^{N} \mathbb{K}^n \phi(t) = \int_0^t \sum_{n=1}^{N} K_n(t,s)\,\phi(s)\,ds \xrightarrow[\text{abs.}]{N\to\infty} \int_0^t \sum_{n=1}^{\infty} K_n(t,s)\,\phi(s)\,ds \qquad \text{(pointwise)}$$

The (absolute) convergence of operators in item (d), applied to $\phi \in L^2(I)$, gives

$$\sum_{n=1}^{N} \mathbb{K}^n \phi \xrightarrow[\text{abs.}]{N\to\infty} \sum_{\substack{n=1 \\ L^2}}^{\infty} \mathbb{K}^n \phi \qquad (\text{in } L^2(I))$$

Since $\sum_{n=1}^{N} \mathbb{K}^n \phi$ converges both pointwise a.e. and in $L^2(I)$ as $N \to \infty$, the limits are equal, i.e.,

$$\sum_{n=1}^{\infty} \mathbb{K}^n \phi(t) = \int_0^t \sum_{n=1}^{\infty} K_n(t,s)\,\phi(s)\,ds$$

Thus, the integral kernel for $\mathbb{H} := \sum_{n=1}^{\infty} \mathbb{K}^n$ is $\sum_{n=1}^{\infty} K_n(t,s) =: H(t,s)$

and $(\mathbb{E} - \mathbb{K})(\mathbb{E} + \mathbb{H}) = \sum_{n=0}^{\infty} \mathbb{K}^n - \sum_{n=1}^{\infty} \mathbb{K}^n = \mathbb{E}$.

So we have proved that an operator $\mathbb{E} - \mathbb{K}$, with $\mathbb{K}$ of Volterra type is always invertible, and that its inverse is again of the form $\mathbb{E} + \mathbb{H}$, with $\mathbb{H}$ of Volterra type. Also, $\mathbb{H}$ is obtained from $\mathbb{K}$ as a Neumann sum.

Thus the equation $\phi - \mathbb{K}\phi = f \in L^2(I)$ has a unique solution in $L^2(I)$, $\phi = f + \mathbb{H}f$.