Math 7320 @ LSU
Spring, 2021
Problem Set 1

1. Prove that $e^{A+B} = e^A e^B$ if $AB = BA$; and prove that

$$e^{t(A+B)} = e^{tA}\, e^{tB} \qquad \text{for all } t \in \mathbb{R}$$

if and only if $AB = BA$.

2. Putzer's theorem provides an algorithm for computing the solution of a linear constant-coefficient homogeneous system $\dot{x} = Ax$ without first computing the Jordan normal form for the generator A. Let A be a linear operator in a finite-dimensional complex vector space of dimension n, and let $\{\lambda_i\}_{i=1}^n$ denotes its eigenvalues. For $k = 1, \ldots, n$, define the operators

$$A_k = \prod_{j=1}^{k} (A - \lambda_j E)$$

and the scalar functions $r_j(t)$ as the solution of the system

$$\begin{aligned} dr_1/dt &= \lambda_1 r_1\,, & r_1(0) &= 1 \\ dr_j/dt &= \lambda_j r_j + r_{j-1}\,, & r_j(0) &= 0 \quad \text{for } j \geq 2. \end{aligned}$$

Putzer's theorem states that

$$e^{tA} = \sum_{k=0}^{n-1} r_{k+1}(t) A_k.$$

Prove Putzer's theorem.

3. Let $\mathcal{A} := \{A(t) : t \in (a,b)\}$ be a commuting invertible family of $n \times n$ matrices with complex entries that is differentiable with respect to t. Define

$$\begin{aligned} \mathcal{A}^{-1} &:= \{A(t)^{-1} : t \in (a,b)\} \\ \dot{\mathcal{A}} &:= \{\dot{A}(t) : t \in (a,b)\}, \end{aligned}$$

in which the dot refers to differentiation with respect to t. Prove that the union $\mathcal{A} \cup \mathcal{A}^{-1} \cup \dot{\mathcal{A}}$ is a commuting family of matrices.

4. Let J be a finite interval in the real line, and let B be the ball $B = \big\{ y \in \mathbb{R}^n \;\big|\; |y - x_0| \leq \rho \big\}$, where $x_0 \in \mathbb{R}^n$ and $\rho > 0$ are fixed. Prove that the set of functions

$$X = \big\{ x : J \to B \;\big|\; x \text{ is continuous} \big\}$$

endowed with the supremum norm

$$\|x_1 - x_2\|_{\sup} = \sup_{t \in J} |x_1(t) - x_2(t)|$$

is a complete metric space.

5. Let X be defined as in problem (4), let t_0 be in J, and let $F : B \to \mathbb{R}^n$ be of Lipschitz class. Prove that, for each function $x \in X$, the following two conditions are equivalent.

(IVP) x is differentiable at each point of J, $\dot{x} = F(x)$, and $x(t_0) = x_0$.

(IE) $x(t) = x_0 + \int_{t_0}^{t} F(x(s))\, ds\,.$