Math 7320 @ LSU
Spring, 2021
Problem Set 4

1. Consider the linear non-autonomous nonhomogeneous linear n^{th}-order initial-value ordinary differential equation

$$\begin{aligned} & x^{(n)}(t) + \sum_{\ell=1}^{n} a_\ell(t)\, x^{(n-\ell)}(t) = F(t) \\ & x^{(\ell)}(t_0) = c_\ell\,, \quad (0 \le \ell < n) \end{aligned} \tag{1}$$

Set $\phi(t) = x^{(n)}(t)$. Prove that, if $x(t)$ satisfies (1), then $\phi(t)$ satisfies the Volterra integral equation

$$\phi(t) + \int_{t_0}^{t} K(t,s)\phi(s) \;=\; f(t)\,,$$

in which the kernel K is defined by

$$K(t,s) \;=\; \sum_{\ell=1}^{n} a_\ell(t)\frac{(t-s)^{\ell-1}}{(\ell-1)!}$$

and the term of inhomogeneity is

$$f(t) = F(t) - \sum_{\ell=1}^{n} a_\ell(t)\left(c_{n-1}\frac{(t-t_0)^{\ell-1}}{(\ell-1)!} + \cdots + c_{n-\ell} \right).$$

2. Consider the spectral Volterra-integral problem

$$\phi(t) \;-\; \lambda\int_{t_0}^{t} K(t,s)\phi(s)\,ds \;=\; f(t) \qquad (t_0 \le t \le t_1),$$

in the L^2 theory. Prove that, for fixed f, the solution ϕ is analytic in λ and satisfies a Volterra integral equation

$$\phi(t) \;=\; f(t) - \lambda\int_{t_0}^{t} H(t,s;\lambda)f(s)\,ds \qquad (t_0 \le t \le t_1).$$

Find the kernel $H(t,s;\lambda)$ in terms of the kernel $K(t,s)$.

(More on the next page ...)

3. Consider the initial-value problem on the interval $I = (t_0 - a, t_0 + a)$

$$\begin{aligned} \frac{d}{dt}x(t) &= A(t)x(t) + f(t)\,, \qquad t \in I \\ x(t_0) &= x_0\,, \end{aligned}$$

in which $A \in L^2(I)$ and $f \in L^1(I)$. Prove that this problem has a unique solution in $L^2(I)$ and that this solution is absolutely continuous.