Math 7320 @ LSU
Spring, 2021
Problem Set 5

1. Let the transfer matrix for the ordinary differential equation

$$\frac{d}{dt}x(t) \;=\; \big(A_0(t) + \sigma A_1(t)\big)x(t)$$

be denoted by $X_\sigma(t,t_0)$. This means that the solutions satisfy

$$x(t) \;=\; X_\sigma(t,t_0)\,x(t_0).$$

Prove that

$$X_\sigma(t,t_0) \;=\; X_0(t,t_0) \,+\, \sigma\int_{t_0}^{t} X_0(t,s)A_1(s)X_0(s,t_0)ds + O(\sigma^2) \qquad (\sigma \to 0).$$

Prove this rigorously, paying attention to the precise definition of the "big O" symbol $O(\sigma)$.

2. Consider the initial-value problem

$$\begin{aligned} -u'' + q(x)u \;&=\; f(x)\,, \\ u(0) = a, \quad u'(0) &= b\,, \end{aligned} \qquad x \in I = [0,\pi], \tag{1}$$

in which $f \in L^1(I)$ and $q \in L^2(I)$. Prove that this problem has a unique absolutely continuous solution that satisfies

$$u(x) - \int_0^x (x-y)q(y)u(y)\,dy \;=\; a + b\,x - \int_0^x (x-y)f(y)\,dy\,.$$

Prove that, given $C > 0$, there exist real numbers M and N such that, for all $q \in L^2(I)$,

$$\|q\|_2 \,<\, C \;\Longrightarrow\; \|u\|_2 \,<\, M$$

and

$$\|q\|_2 \,<\, C \;\Longrightarrow\; \max_{x\in I}|u(x)| \,<\, N.$$

3. Consider the initial-value problem

$$\begin{aligned} -u_\epsilon'' + \big(q(x)+\epsilon\big)u_\epsilon \;&=\; 0\,, \\ u_\epsilon(0) = a, \quad u_\epsilon'(0) &= b\,, \end{aligned} \qquad x \in I = [0,\pi],$$

in which a and b are fixed complex numbers and ϵ is a complex variable.

a. Prove that, for each $\rho > 0$, there are numbers M and N such that, for all $\epsilon \in \mathbb{C}$,

$$|\epsilon| < \rho \;\Longrightarrow\; \|u_\epsilon\|_2 < M$$

and

$$|\epsilon| < \rho \implies \max_{x \in I} |u_\epsilon(x)| < N.$$

b. By deriving an initial-value problem of the form (1) for the function $u_\epsilon - u_0$, prove that the map $\epsilon \mapsto u_\epsilon \in L^2(I)$ is a continuous function.

c. By deriving an initial-value problem of the form (1) for the function $(u_\epsilon - u_0)/\epsilon$, use the causal Green function for the operator $-d^2/dx^2 + q(x)$ to prove that, for each $x \in I$, $u_\epsilon(x)$ is differentiable with respect to the complex variable ϵ at $\epsilon = 0$.

4. **Note.** (This is not part of the assignment.) One can prove that, for the solution $u_\epsilon(x)$ of the previous initial-value problem, the derivatives $u'_\epsilon(x)$ are analytic in ϵ at $\epsilon = 0$. Here is one way to do this. Using the DE, write an integral expression for $u'(x)$ that involves u. Using the results of previous problem, this will allow you to bound $u'(x)$ pointwise, even over all ϵ in any bounded set. Then use the analyticity of $u_\epsilon(x)$ (for *all* ϵ) and a theorem from complex analysis, *etc.*