Existence and uniqueness

We will do local existence and uniqueness theory for the nonlinear initial-value problem

(IVP) $$\begin{cases} \dot{x} = f(x) \\ x(t_0) = \bar{x} \end{cases}$$

First, some definitions and facts. I will assume that you know about metric spaces and normed vector spaces.

Defn A metric space (X, d) is bounded if $\exists\, M \in \mathbb{R}$ s.th. $\forall\, x_1, x_2 \in X$, $d(x_1, x_2) < M$.

Defn Let (X, d_X) and (Y, d_Y) be metric spaces. A function $f: X \to Y$ is bounded if $\exists\, y \in Y$ and $M \in \mathbb{R}$ s.th. $\forall\, x \in X$, $d_Y(f(x), y) < M$. f is Lipschitz with constant $K \in \mathbb{R}$ if $\forall\, x_1, x_2 \in X$, $d_Y(f(x_1), f(x_2)) \leq K d_X(x_1, x_2)$

Defn Let (X, d_X) be a metric space. A function $f: X \to X$ is a contraction on X if f is Lipschitz with constant K satisfying $0 \leq K < 1$.

Fact (exercise) If a map $f: X \to Y$ of metrix spaces is Lipschitz, then f is continuous. If, additionally, X is bounded, then f is bounded.

Given positive numbers a and ρ, solutions of (IVP) will be sought in the set of continuous functions $x : J_a \to B_\rho$, where

$$J_a = \{ t \in \mathbb{R} : |t - t_0| < a \}$$
$$B_\rho = \{ y \in \mathbb{R}^n : |y - \bar{x}| < \rho \}$$

This set will be denoted by $\mathcal{X}$:

$$\mathcal{X} = \{ x : J_a \to B_\rho \mid x \text{ is continuous} \}.$$

$\mathcal{X}$ becomes a metric space when endowed with the following distance:

$$\| x_1 - x_2 \|_{sup} = \sup_{t \in J_a} | x_1(t) - x_2(t) | .$$

You can prove this by the following steps:

(1) The vector space $BC(J_a, \mathbb{R}^n)$ of bounded continuous functions from J_a to $\mathbb{R}^n$, with supremum norm $\|x\|_{sup} = \sup_{t \in J_a} |x(t)|$ is a Banach space (complete normed vector space).

(2) $\|x - y\|_{sup}$ defines a metric on any subset of $BC(J_a, \mathbb{R}^n)$.

(3) B_ρ being closed (and bounded) makes X complete in this metric.

<u>Theorem</u> Let $f : B_\rho \to \mathbb{R}^n$ be Lipschitz with constant K, and set $M = \sup_{y \in B_\rho} |f(y)|$. If $a \in \mathbb{R}$ is such that $0 < a < \{ \rho M^{-1}, K^{-1} \}$, then there exists a unique function $x \in \mathcal{X}$ that satisfies (IVP).

<u>Proof</u> First, prove that, for each $x \in \mathcal{X}$, the following two conditions are equivalent.

$$\text{(IVP)} \quad \dot{x} = f(x) \quad \text{and} \quad x(t_0) = \bar{x}$$

$$\text{(IE)} \quad x(t) = \bar{x} + \int_{t_0}^{t} f(x(s))\,ds,$$

in which (IE) stands for "integral equation". In (IVP), differentiability of x is tacitly part of the condition. Details are left as an exercise.

Define a map $\mathcal{F} : \mathcal{X} \to (\mathbb{R}^n)^{J_a}$

$$(\mathcal{F}x)(t) = \bar{x} + \int_{t_0}^{t} f(x(s))\,ds \qquad \forall\ t \in J_a.$$

Let us prove that $\mathcal{F}$ is a contraction on $\mathcal{X}$. Given $x \in \mathcal{X}$, $f \circ x : J_a \to \mathbb{R}^n$ is also continuous, so that $(\mathcal{F}x)(t)$ is well defined and continuous and

$$|\mathcal{F}x(t) - x_0| \le \int_{t_0}^{t} |f(x(s))|\,ds \le aM < \rho,$$

and therefore $\mathcal{F}x \in \mathcal{X}$. For all $x_1, x_2 \in \mathcal{X}$,

$$(\mathcal{F}x_1 - \mathcal{F}x_2)(t) = \int_{t_0}^{t} \left[f(x_1(s)) - f(x_2(s)) \right] ds,$$

and thus, by the Lipschitz property of f, $\forall\ t \in J_a$

$$|(\mathcal{F}x_1 - \mathcal{F}x_2)(t)| \le \int_{t_0}^{t} |F(x_1(s)) - F(x_2(s))|\,ds$$

$$\le \int_{t_0}^{t} K\,|x_1(s) - x_2(s)|\,ds \le aK\,\|x_1 - x_2\|_{\sup}.$$

Therefore $\|\mathcal{F}x_1 - \mathcal{F}x_2\|_{sup} \leq aK\|x_1 - x_2\|_{sup}$.

Since $aK < 1$, $\mathcal{F}$ is indeed a contraction in $\mathcal{X}$.

Now notice that $\forall x \in \mathcal{X}$, $(\mathcal{F}x)(t_0) = \bar{x}$, and so (IE) is equivalent to

(FP) $\mathcal{F}: \mathcal{X} \to \mathcal{X}$ admits a fixed point.

Indeed, (IE) is exactly the statement that $x \in \mathcal{X}$ and $\mathcal{F}x = x$. To prove existence of a fixed point, choose $x_0 \in \mathcal{X}$ with $x_0(t_0) = \bar{x}$, and define recursively $x_{n+1} = \mathcal{F}x_n$ for $n \geq 1$, that is,

$$x_{n+1}(t) = \bar{x} + \int_{t_0}^{t} f(x_n(s))\,ds .$$

Inductively, one proves by the contraction property of $\mathcal{F}$ that

$$\|x_{n+1} - x_n\|_{sup} \leq (aK)^n \|x_1 - x_0\|_{sup} .$$

Since $aK < 1$, the exponential convergence of $\|x_{n+1} - x_n\|_{sup}$ to zero makes $\{x_n\}_{n=0}^{\infty}$ a Cauchy sequence in $\mathcal{X}$. Since $\mathcal{X}$ is complete, this sequence admits a limit $x \in \mathcal{X}$, and it is a fixed point of $\mathcal{F}$ because

$$\mathcal{F}x = \mathcal{F}\lim_{n\to\infty} x_n = \lim_{n\to\infty} \mathcal{F}x_n = \lim x_n = x$$

Let $y \in \mathcal{X}$ be a fixed point of $\mathcal{F}$. Then

$$\|x-y\| = \|\mathcal{F}x - \mathcal{F}y\| \le aK\,\|x-y\|,$$

and since $aK < 1$, we obtain $\|x-y\| = 0$ so that $x = y$. Thus the fixed point of $\mathcal{F}$ is unique. ∎

Some consequences of the existence and uniqueness theorem:

- Given a solution $x : t_0 + (-a,a) \to B_\rho$ with $x(t_0) = \bar{x}$, the function $y : t_1 + (-a,a) \to B_\rho$ defined by

$$y(t) = x(t + t_0 - t_1)$$

satisfies $\dot{y} = f(y)$ and $y(t_1) = \bar{x}$.

- Solutions can't cross: If $\dot{x} = f(x)$ and $\dot{y} = f(y)$ and $x(t_0) = y(t_1)$, then $x(t_0 + s) = y(t_1 + s)$ on any s-interval on which both of these are defined.

- Periodic solutions: If $x : (a,b) \to \mathbb{R}^n$ satisfies $\dot{x} = f(x)$ and $x(t_0) = x(t_0 + p)$ for some $t_0 \in (a,b)$ and p such that $t_0 + p \in (a,b)$, then x can be extended uniquely to a periodic function $x : \mathbb{R} \to B_\rho$ that satisfies $\dot{x} = f(x)$ and $x(t+p) = x(t)$ $\forall\, t \in \mathbb{R}$.

Nonautonomous systems:

Let $f: t_0+(-b,b) \times B_\rho \longrightarrow \mathbb{R}^n$ be Lipschitz continuous with constant K, and let f be bounded by $M < \infty$. Then there is an interval $t_0+(-a,a)$ such that the IVP

$$\begin{cases} \dot{x} = f(t,x) \\ x(t_0) = \bar{x} \end{cases}$$

admits a unique solution $x: t_0+(-a,a) \longrightarrow B_\rho$.

You can prove this by applying the theorem for autonomous systems to the extended autonomous system

$$\begin{cases} \dot{x} = f(\theta,x) & x(t_0) = \bar{x} \\ \dot{\theta} = 1 & \theta(t_0) = t_0 \end{cases}.$$

To show that $a < \min\{K^{-1}, \rho M^{-1}, b\}$ works, you have to do a little bit more than directly applying the previous theorem.

Lemma (Gronwall's inequality)

Let $f: [0,\alpha] \to \mathbb{R}$ be a continuous function with non-negative values, and let $C, K > 0$ be such that

$$f(t) \le C + \int_0^t K f(s)\, ds \quad \forall \quad t \in [0,\alpha].$$

Then $f(t) \le Ce^{Kt} \quad \forall \ t \in [0,\alpha]$.

<u>Proof</u> For $C>0$, set $g(t) := C + \int_0^t K f(s)\,ds$. Then $g'(t) = Kf(t)$, so that $g'(t)\,g(t)^{-1} \le K$

Thus $\log g(t) = \log g(0) + \int_0^t g'(s) g(s)^{-1} ds \le \log C + Kt$

and so $g(t) \le Ce^{Kt}$ and $\therefore$ $f(t) \le Ce^{Kt}$

For $C=0$, one has $f(t) \le \tilde{C} + \int_0^t Kf(s)\,ds$ $\forall$ $\tilde{C}>0$, and thus $f(t) \le \tilde{C}e^{Kt}$ $\forall$ $\tilde{C}>0$, so that $f(t) \le 0e^{Kt}$. ∎

<u>Theorem</u> (continuous dependence on initial conditions) Let $x_1(t)$ and $x_2(t)$ be solutions to $\dot{x} = f(x)$ on $t_0 + (-a,a)$. Then

$$|x_1(t) - x_2(t)| \le |x_1(t_0) - x_2(t_0)|\, e^{K|t-t_0|}$$

for all $t \in t_0 + (-a,a)$, where K is a Lipschitz constant for f.

<u>Proof</u> The case $t<t_0$ can be converted to the case $t>t_0$ by replacing $f(x)$ with $-f(x)$, which has the same Lipschitz constant as f. Assume now that $t \ge t_0$.

For $i \in \{1,2\}$,

$$x_i(t) = x_i(t_0) + \int_{t_0}^t f(x_i(s))\,ds$$

So $x_1(t) - x_2(t) = x_1(t_0) - x_2(t_0) + \int_{t_0}^t [f(x_1(s)) - f(x_2(s))]\,ds$,

and
$$\begin{aligned}|x_1(t) - x_2(t)| &\le |x_1(t_0) - x_2(t_0)| + \int_{t_0}^t |f(x_1(s)) - f(x_2(s))|\,ds \\ &\le |x_1(t_0) - x_2(t_0)| + \int_{t_0}^t K|x_1(s) - x_2(s)|\,ds.\end{aligned}$$

Gronwall's inequality now yields the result. ▨

This theorem implies that the flow $\phi_t(x)$ at any time t is a continuous function of the initial state x. It can be applied to obtain continuous dependence of flows on parameters of the system.

Theorem Let $f: \mathbb{R}^n \times \mathbb{R}^k \to \mathbb{R}^n$ be Lipschitz with constant K. Let $x_1(t)$ and $x_2(t)$ satisfy the following IVPs in $\mathbb{R}^n$:

$$\dot{x}_1 = f(x_1, y_1^0), \quad x_1(0) = x_1^0$$
$$\dot{x}_2 = f(x_2, y_2^0), \quad x_2(0) = x_2^0 .$$

Then

$$|x_1(t) - x_2(t)| \leq \left[|x_1^0 - x_2^0| + |y_1^0 - y_2^0| \right] e^{K|t-t_0|} .$$

As a result, the flow $\phi_t(x,y)$ in $\mathbb{R}^n$ is a continuous function of the initial condition x, the parameter y, and time t.

Proof The theorem is a direct application of the previous theorem to the ODE in $\mathbb{R}^n \times \mathbb{R}^k$ given by

$$(x,y)' = g(x,y) := (f(x), 0)$$

with initial conditions (x_1^0, y_1^0) and (x_2^0, y_2^0). ▨

Assume now that f is continuously differentiable.
We will derive the derivative of the trajectory of $\dot{x} = f(x)$ as the initial condition is varied.

Let $x(t,\xi)$ satisfy $x(t,\xi) \in B_\rho$ and $|\xi| < \varepsilon$ \$ $\varepsilon > 0$ and

$$\begin{cases} \dot{x}(t,\xi) = f(x(t,\xi)), & t \in (-a,a) \\ x(0,\xi) = x_0 + \xi, \end{cases}$$

where the dot indicates differentiation with respect to the first variable t. Let $u(t,\xi)$ satisfy the linear ODE IVP (for $|\xi| < \varepsilon$)

$$\begin{cases} \dot{u}(t,\xi) = Df_{x(t,0)}\, u(t,\xi), & t \in (-a,a) \\ u(0,\xi) = \xi. \end{cases}$$

Define $g(t,\xi) = \left| x(t,\xi) - x(t,0) - u(t,\xi) \right|$.

We will prove that

$$\frac{g(t,\xi)}{|\xi|} \longrightarrow 0 \quad \text{as} \quad \xi \to 0.$$

From the IVPs that $x(t,\xi)$ and $u(t,\xi)$ satisfy, we obtain

$$(*) \qquad g(t,\xi) \le \int_0^t \left| f(x(s,\xi)) - f(x(s,0)) - Df_{x(s,0)}\, u(s,\xi) \right| ds$$

By the differentiability of f, we have

(+) $$f(x(s,\xi)) - f(x(s,0)) = Df_{x(s,0)}\big(x(s,\xi) - x(s,0)\big) + R(s,\xi)\,\big|x(s,\xi) - x(s,0)\big|,$$

in which $R(s,\xi) \to 0$ as $x(s,\xi) - x(s,0) \to 0$.
In fact, this convergence is uniform over s because Df is uniformly continuous on B_ρ. It is left as an exercise to prove this.

From the theorem on continuous dependence on initial conditions, we obtain

$$|x(s,\xi) - x(s,0)| \le |\xi| e^{K|s|} \le |\xi| e^{aK}.$$

Thus $R(s,\xi) \to 0$ as $\xi \to 0$ uniformly in s.
Putting $\tilde{R}(s,\xi) = R(s,\xi)e^{K|s|}$, and using (+) in (*), we obtain

$$g(t,\xi) \le \int_0^t \big| Df_{x(s,0)}(x(s,\xi) - x(s,0) - u(s,\xi))\big|\,ds + |\xi| \int_0^t |\tilde{R}(s,\xi)|\,ds.$$

Since $\tilde{R}(s,\xi) \to 0$ as $\xi \to 0$, uniformly in s,

$$\int_0^t |\tilde{R}(s,t)|\,ds \to 0 \quad \text{as } \xi \to 0.$$

Let $|Df_y| \le N \ \forall\, y \in B_\rho$. Then, since f is C^1 on B_ρ,

$$g(t,\xi) \le \int_0^t N g(s,\xi)\,ds + |\xi| \int_0^t |\tilde{R}(s,\xi)|\,ds$$

Gronwall's inequality gives $g(t,\xi) \le |\xi| e^{Nt} \int_0^t |\tilde{R}(s,\xi)|\,ds$,

(++) so $\dfrac{g(t,\xi)}{|\xi|} \to 0$ as $\xi \to 0$ uniformly in $t \in (-a,a)$.

Since $u(t,\xi)$ satisfies a linear ODE, the map from initial condition to solution, $\xi \longmapsto u(\cdot,\xi)$ is linear [Recall $u(t,\xi) = U(t)\xi$, where $U(t)$ is a matrix solution with $U(0) = E$.], and it is, by virtue of $(**)$, equal to the derivative of the trajectory $x(t,\xi)$ of $\dot{x} = f(x)$ as a function of the initial condition, at x_0.

Notice two special situations:

(1) If x_0 is a fixed point of the flow, that is, $f(x_0) = 0$, so $x(t,0) \equiv x_0$, then the system for $u(t,\xi)$ is autonomous:

$$\begin{cases} \dot{u}(t,\xi) = Df_{x_0}\, u(t,\xi) \\ u(0,\xi) = \xi, \end{cases}$$

so the derivative of the trajectory of $\dot{x} = f(x)$ about x_0 is the exponential flow

$$u(t,\xi) = \exp\left(t\, Df_{x_0}\right)\xi .$$

(2) If $x(t,0) = x_0$ for some $t > 0$, then $x(t,0)$ is extended to all $t \in \mathbb{R}$ as a periodic orbit. (assume x_0 is not a fixed point). Now, $u(t,\xi)$ satisfies a non-autonomous linear ODE with periodic coefficients.