General Stuff on ODES

A first-order n-dimensional ordinary differential equation (ODE) is an expression of the form

(ODE) $$\dot{x} = f(x, t),$$

with $f: \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ being a t-dependent vector field in $\mathbb{R}^n$. A solution of the ODE is a function $x: (a,b) \to \mathbb{R}^n$ such that

solution = trajectory = flow...

$$\frac{dx}{dt}(t) = f(x(t), t) \quad \forall\ t \in (a,b),$$

where we tacitly assume that $x(t)$ is differentiable. The initial-value problem (IVP) is

(IVP) $$\dot{x} = f(x,t) \ ; \quad x(t_0) = x_0 .$$

The ODE is autonomous if f is independent of t, so that

$$\dot{x} = f(x).$$

An m^{th}-order n-dimensional ODE of the form

$$x^{(m)} = F(x^{(m-1)}, \ldots, x', x, t)$$

can be converted into a first-order $(mn+1)$-dimensional autonomous ODE $\dot{y} = f(y)$ with dependent variable

$$y = (x_{m-1}, x_{m-2}, \ldots, x_1, x, \theta)$$

and $f(y) = \big(F(x_{m-1}, \ldots, x_1, x, \theta), x_{m-1}, \ldots, x_2, x_1, 1\big)$,
which, when written out, reads

$$\begin{cases} \dot{x} = x_1 \\ \dot{x}_1 = x_2 \\ \vdots \\ \dot{x}_{m-2} = x_{m-1} \\ \dot{x}_{m-1} = F(x_{m-1}, \ldots, x_1, x, \theta) \\ \dot{\theta} = 1 \end{cases}$$

If $x(t)$ is a solution of $\dot{x} = f(x,t)$ (on some t-interval), then the function defined by $y(t) = x(-t)$ satisfies

$$\dot{y} = -f(y, -t)$$

(verify this); and $z(t) = x(t-t_0)$ satisfies $\dot{z} = f(z, t-t_0)$. Thus autonomous ODEs are t-shift-invariant.

A <u>linear</u> ODE is of the form

$$\dot{x} = A(t)x + g(t),$$

where $\forall t$, $A(t)$ is a linear transformation of $\mathbb{R}^n$ and g takes values in $\mathbb{R}^n$. If $g(t) \equiv 0$, then it is a homogeneous linear ODE, and if $A(t) \equiv A$, then it is a homogeneous linear ODE with <u>constant coefficients</u>.

<u>Flows of vector fields</u>. Given a vector field $f : \mathbb{R}^n \to \mathbb{R}^n$ and the associated ODE $\dot{x} = f(x)$, we sometimes use the following notation for the solution $x(t)$ with $x(0) = y$:

(assuming existence uniqueness of soln.)

$$x(t) = \phi_t(y).$$

Thus $\frac{d}{dt}\phi_t(y) = f(\phi_t(y))$ and $\phi_0(y) = y$.

From this point of view, at each time t, ϕ_t is a transformation of $\mathbb{R}^n$, flowing any initial value y to the point on its trajectory at time t.
Flows of vector fields occur on differentiable manifolds, more generally, particularly in geometric mechanics, where the manifold is the configuration space of, say, a robot.

[Diversion] States vs. "observables" and "global linearization".
For an ODE $\dot{x} = f(x)$, the value of $x \in \mathbb{R}^n$ is called the <u>state</u> of the system. Von Neumann and Koopman developed an approach to studying flows that keeps track of measurements at all states (observables) rather than the states themselves. Think of an observable as a function

$$V : \mathbb{R}^n \to \mathbb{R}$$

that assigns a real value (or an element of a vector space, more generally) to each state $x \in \mathbb{R}^n$.

Given an ODE $\dot{x} = f(x)$ and an observable $V \in C^1(\mathbb{R}^n)$, define $\dot{V}$ to be the derivative of V along the flow of f:

$$\dot{V}(x) := \frac{d}{dt} V(\phi_t(x))\Big|_{t=0} = \underbrace{DV}_{\nabla V}(x) \cdot f(x)$$

Now think of this as an ODE of infinite dimension with the (new) states are the observables of the old states, and they live in $C^1(\mathbb{R}^n)$ (instead of $\mathbb{R}^n$). The IVP is

(K) $$\dot{V} = DV \cdot f$$

and the solution is $V[t] \in C^1(\mathbb{R}^n)$, where

$$V[t](x) = V(\phi_t(x)),$$

with initial condition $V[0] = V$, or, more explicitly, $V[0](x) = V(\phi_0(x)) = V(x)$. Notice that (K) is a linear ODE in $C^1(\mathbb{R}^n)$ ~ this is "global linearization", or "Koopmanism". Its flow $\Phi_t(V) = V(\phi_t(\cdot))$ is the Koopman operator.

We won't study this, but I think it's good to know of its existence. In our department at LSU, it is studied by F. Neubrander.

End of diversion

We will see observables in the form of Lyapunov functions, though.

Examples of famous ODEs

- Stationary 1D nonlinear Schrödinger equation

$$-\frac{d^2}{dx^2}\psi + \underbrace{V(x)\psi}_{\text{linear part}} + \underbrace{|\psi|^2\psi}_{\text{nonlinear part}} = E\psi$$

where E is a fixed complex number.

- $\ddot{x} + \delta\dot{x} - \alpha x + \beta x^3 = \gamma\cos\omega t$

  $\delta \geq 0, \ \alpha, \beta > 0$

  This is the Duffing oscillator $\leadsto$ chaos

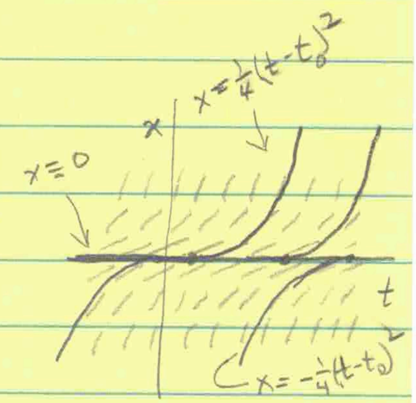
- $\ddot{x} + \mu(x^2 - 1)\dot{x} + x = 0, \qquad \mu > 0$

  Van der Pol oscillator $\leadsto$ periodic orbits

- The Lorenz system $\leadsto$ chaos and strange attractors

$$\begin{cases} \dot{x} = \sigma(y - x) \\ \dot{y} = -xz + \rho x - y \\ \dot{z} = xy - \beta z \end{cases}$$

Example of nonuniqueness in 1D. $\qquad \dot{x}(t) = \sqrt{|x(t)|}$



Existence for only finite time in 1D

$$\begin{cases} \dot{x}(t) = f(x(t)) \\ x(0) = y \end{cases} \implies t = \int_y^{x(t)} \frac{dz}{f(z)}$$

If $\int_y^{\infty} \frac{dz}{f(z)} < \infty$, then there is "finite-time blowup"

e.g. $f(x) = x^{1+\varepsilon}$ for $x > 0$ $(\varepsilon > 0)$