Linear homogeneous autonomous ODEs

$$\dot{x} = Ax \qquad (x \in \mathbb{R}^n,\ A : \mathbb{R}^n \to \mathbb{R}^n \text{ linear})$$

Uniqueness Let $x_1(t)$ and $x_2(t)$ be solutions of $\dot{x} = Ax$ for all $t \in (a,b)$ such that $x_1(t_0) = x_2(t_0)$ for some $t_0 \in (a,b)$. Then $x_1(t) = x_2(t)$ $\forall\ t \in (a,b)$. [Note $a = -\infty$ or $b = \infty$ is allowed.]

Proof For $t \in (a,b)$, set $x(t) = x_1(t) - x_2(t)$, and define $x(a)$ and $x(b)$ by continuity. Then $\dot{x} = Ax$ and $x(t_0) = 0$. Let $t_1 = \max_{t \in [a,b]} \{x(t) = 0\}$. (This max exists since x is continuous and the set is nonempty.) For all r and t such that $t_1 \le r \le t \le b$, $x(r) = A\int_{t_1}^{r} x(s)\,ds$ and thus

$$|x(r)| \le |A|\,|r - t_1| \max_{s \in [t_1, r]} |x(s)| \le |A|\,|t - t_1| \max_{s \in [t_1, t]} |x(s)|$$

$$\Rightarrow \max_{r \in [t_1, t]} |x(r)| \le |A|\,|t - t_1| \max_{s \in [t_1, t]} |x(s)|$$

$$\Rightarrow \left(1 - |A|\,|t - t_1|\right) \max_{s \in [t_1, t]} |x(s)| \le 0$$

Thus if $|t - t_1| < |A|^{-1}$, then $\max_{s \in (t_1, t)} |x(s)| = 0$, and by defn of t_1, we must have $t_1 = t$, and thus $t_1 = b$.
An analogous argument proves that $a = \min_{t \in [a,b]} \{x(t) = 0\}$.
We conclude that $x(t) = 0$ $\forall\ t \in [a,b]$, that is $x_1 = x_2$. ∎

<u>Existence</u> There exists a solution $x(t)$ to $\{\dot{x} = Ax,\ x(t_0) = x_0\}$ defined for all $t \in \mathbb{R}$.

E = identity matrix

<u>Proof</u> The matrix IVP $\{\dot{X} = AX,\ X(0) = E\}$ has solution

$$X(t) = \exp(tA) := \sum_{n=0}^{\infty} \frac{1}{n!} t^n A^n$$

Indeed, $\forall\ T$, this series converges uniformly and absolutely for $|t| \le T$; and term-by-term differentiation yields

$$\frac{d}{dt} e^{tA} = Ae^{tA}.$$

The vector function $x(t) = e^{tA} x_0$ therefore satisfies $\dot{x} = Ax$ and $x(0) = Ex_0 = x_0$. ∎

Basic <u>properties</u> of e^{tA}

- $Me^A M^{-1} = e^{MAM^{-1}}$

- $AB = BA \implies e^{A+B} = e^A e^B$

- $\det e^A = e^{\operatorname{tr} A} = e^{\sum_{i=1}^m \lambda_i} = \prod_{i=1}^m e^{\lambda_i}$ [λ_i are the evals of A]

- Since $\det e^{tA} = e^{t \operatorname{tr} A}$ and the flow for $\dot{x} = Ax$ is $\phi_t(x) = e^{tA} x$, the flow scales volumes exponentially with rate $\operatorname{tr} A$. Volume is preserved by the flow if and only if $\operatorname{tr} A = 0$.

Behavior of e^{tA} : normal forms

Suppose that $AM = M\Lambda$. [If A is diagonalizable, Λ could be a diagonal matrix of eigenvalues of A and M the matrix whose columns are the eigenvectors.] Then for $\dot{x} = Ax$,

$$\phi_t(x) = e^{tA}x = e^{tM\Lambda M^{-1}}x = Me^{t\Lambda}M^{-1}x$$

Note that $M^{-1}x$ is the vector of components of x in the basis given by the columns of M.

So we should find $e^{t\Lambda}$ for all normal forms of matrices. Normal forms have canonical blocks :

$$A = \sum_{j=1}^{N} \Lambda_j P_j , \quad E = \sum_{j=1}^{N} P_j , \quad P_j^2 = P_j$$

The second two say that the P_j are complementary projections. The Λ_j are the canonical matrices

If we allow complex solutions to e^{tA}, all the Λ_j are of Jordan form:

$$\Lambda = \lambda E_m + N_m = \begin{bmatrix} \lambda & 1 & 0 & \cdots & 0 \\ 0 & \lambda & 1 & & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ & & & \ddots & 1 \\ 0 & \cdots & \cdots & 0 & \lambda \end{bmatrix}$$

where E_m is the $m \times m$ identity matrix and N_m is the nilpotent matrix $N_m = \left(\delta_{i+1,j}\right)_{i,j=1}^{m}$.

Since E_m and N_m commute, $e^{t\Lambda} = e^{t\lambda E_m} e^{tN_m}$.

We have $e^{t\lambda E_m} = e^{t\lambda} E_m$ and

$$e^{tN_m} = E_m + tN_m + \frac{t^2}{2} N_m^2 + \dots + \frac{t^{(m-1)}}{(m-1)!} N_m^{m-1}$$

because $N_m^m = 0$. Note that the operation $B \mapsto BN_m$ shifts the columns of B to the right (and the rightmost column of B disappears). Thus

Columns are the "modes" of the system.

$$e^{t\Lambda} = e^{t\lambda} \begin{bmatrix} 1 & t & t^2/2 & \cdots & t^{m-1}/(m-1)! \\ 0 & 1 & t & \ddots & \vdots \\ \vdots & & 1 & \ddots & t^2/2 \\ \vdots & & & 1 & t \\ 0 & \cdots & \cdots & 0 & 1 \end{bmatrix}$$

Now suppose that A is real and that we are interested ultimately in the real solutions. Let $\lambda, \bar{\lambda}$ be a conjugate pair of eigenvalues of algebraic multiplicity 1. There is a basis of real vectors $\{v_1, v_2\}$ for the two-dim'l span of the eigenspaces of $a \pm bi$ together, and respect to which A acts like the rotation-dilation matrix

$\lambda = a + bi$

$$A \sim \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \quad \text{on span}\{v_1, v_2\};$$

more precisely,

$$A[v_1 \ v_2] = [v_1 \ v_2] \begin{bmatrix} a & -b \\ b & a \end{bmatrix}.$$

This normal form can be decomposed as

$$C = \begin{bmatrix} a & -b \\ b & a \end{bmatrix} = aE + bJ,$$

with $J = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$. Notice that $J^2 = -E$.
Using this, you can compute that e^{bJ} is a rotation matrix:

$$e^{bJ} = \begin{bmatrix} \cos b & -\sin b \\ \sin b & \cos b \end{bmatrix} =: R(b)$$

Since E and J commute,

$$e^{tC} = e^{taE} e^{tbJ} = e^{ta} R(tb).$$

Thus the flow of $\dot{x} = Ax$ within $\mathrm{span}\{v_1, v_2\}$ is a combination of exponential and rotational behavior.

If λ has higher multiplicity, then the real normal form of A has Jordan-type blocks of the form

$$\Lambda = E \otimes C + N \otimes E = \begin{bmatrix} a & -b & 1 & 0 & \cdots & 0 & 0 \\ b & a & 0 & 1 & & 0 & 0 \\ 0 & 0 & a & -b & 1 & 0 & \vdots \\ 0 & 0 & b & a & 0 & 1 & \\ \vdots & & & \ddots & & \ddots & \\ & & & & & 1 & 0 \\ & & & & & 0 & 1 \\ 0 & 0 & & & 0 & 0 & a & -b \\ 0 & 0 & \cdots & & 0 & 0 & b & a \end{bmatrix} = \begin{bmatrix} C & E & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & & 0 \\ \vdots & & & & E \\ 0 & \cdots & & 0 & C \end{bmatrix}$$

acting in $\mathbb{R}^m \otimes \mathbb{R}^2$, and

$$e^{t\Lambda} = e^{ta} \begin{bmatrix} R(tb) & tR(tb) & \cdots & \frac{t^{m-1}}{(m-1)!} R(tb) \\ 0 & R(tb) & \ddots & \vdots \\ \vdots & & \ddots & \\ & & & tR(tb) \\ 0 & \cdots & 0 & R(tb) \end{bmatrix}$$