Definitions, given an ODE $\dot{x} = f(x)$ in $\mathbb{R}^n$.

- $\bar{x} \in \mathbb{R}^n$ is a fixed point (equilibrium/stationary solution/trajectory) if $x(t) \equiv \bar{x}$ is a solution, or, equivalently, if $f(\bar{x}) = 0$.

- A (nontrivial) periodic trajectory (orbit) is a solution $x(t)$ for which $\exists\ T > 0$ s.th. $x(t+T) = x(t)\ \forall\ t \in \mathbb{R}$.

- A subset $S \subset \mathbb{R}^n$ is positively (negatively) invariant under the flow if $\forall\ x \in S\ \forall\ t \geq 0\ (\leq 0)$, $\phi_t(x) \in S$.

- Let $\bar{x}$ be a fixed point and $\mathcal{U}$ an open set with $\bar{x} \in \mathcal{U}$.
The local stable manifold is
$$W^s_{loc}(\bar{x}, \mathcal{U}) = \{x \in \mathcal{U} : \phi_t(x) \in \mathcal{U}\ \forall t \geq 0 \text{ and } \phi_t(x) \to \bar{x} \text{ as } t \to \infty\}$$
This is a positively invariant set.
The local unstable manifold is
$$W^u_{loc}(\bar{x}, \mathcal{U}) = \{x \in \mathcal{U} : \phi_t(x) \in \mathcal{U}\ \forall\ t \leq 0 \text{ and } \phi_t(x) \to \bar{x} \text{ as } t \to -\infty\}$$
This is a negatively invariant set.
The global stable manifold is
$$W^s(\bar{x}) = \bigcup_{t \in \mathbb{R}} \phi_t(W^s_{loc}(\bar{x}, \mathcal{U})) = \bigcup_{t \leq 0} \phi_t(W^s_{loc}(\bar{x}, \mathcal{U}))$$
The global unstable manifold is
$$W^u(\bar{x}) = \bigcup_{t \in \mathbb{R}} \phi_t(W^u_{loc}(\bar{x}, \mathcal{U})) = \bigcup_{t \geq 0} \phi_t(W^u_{loc}(\bar{x}, \mathcal{U}))$$
Both of these sets are + and − invariant and they are independent of the open set $\mathcal{U}$ about $\bar{x}$.

- Let $\bar{x}$ be a fixed point, and let $x(t)$ be a trajectory such that $x(t) \to \bar{x}$ as $t \to \infty$ and as $t \to -\infty$. Such a trajectory is called a homoclinic orbit.

- Let $\bar{x}_1$ and $\bar{x}_2$ be different fixed points, and let $x(t)$ be a trajectory such that $x(t) \to \bar{x}_1$ as $t \to -\infty$ and $x(t) \to \bar{x}_2$ as $t \to \infty$. Such a trajectory is called <u>heteroclinic</u>.

- A fixed point $\bar{x}$ is <u>Lyapunov stable</u> (or just <u>stable</u>) if, for each neighborhood V of $\bar{x}$, there is a neighborhood V' of $\bar{x}$ with $V' \subset V$ such that $\forall\, x \in V'$ and $\forall\, t \geq 0$, $\phi_t(x) \in V$.

- A fixed point $\bar{x}$ is <u>asymptotically stable</u> if it is Lyapunov stable and there exists a neighborhood V of x such that $\forall\, x \in V$, $\phi_t(x) \to \bar{x}$ as $t \to \infty$.

- A <u>sink</u> for f is an asymptotically stable fixed pt for f; and a <u>source</u> for f is an asymptotically stable fixed pt. for $-f$.

- Let W be a nbhd of a fixed point $\bar{x}$ and $V : W \to \mathbb{R}$ a differentiable function. V is a <u>Lyapunov function</u> for $\bar{x}$ if
  (i) $V(\bar{x}) = 0$, $V(x) > 0 \;\; \forall\, x \in W \setminus \{\bar{x}\}$
  (ii) $\nabla V(x) \cdot f(x) \leq 0 \;\; \forall\, x \in W \setminus \{\bar{x}\}$.
  V is a <u>strict</u> Lyapunov function if $\nabla V \cdot f\big|_x < 0 \;\; \forall\, x \in W \setminus \{\bar{x}\}$.

- A fixed point $\bar{x}$ is <u>globally asymptotically stable</u> if there is a global strict Lyapunov function for $\bar{x}$ (i.e. $W = \mathbb{R}^n$).

exercise

<u>Lemma</u> (continuation of bounded solutions) Let $f : \overset{\text{open}}{W} \to \mathbb{R}^n$ be Lipschitz, and let $x(t)$ solve $\dot{x} = f(x)$ for all $t \in (a,b)$, and let $M > 0$ be such that $|x(t)| \leq M \;\; \forall\, t \in (a,b)$. Then $\exists\, c, d \in \mathbb{R}$ with $[a,b] \subset (c,d)$ s.th. the soln $x(t)$ can be extended to all $t \in (c,d)$.

& $\bar{B}_M \subset W$

Theorem (GH p 4) Let $f: W \to \mathbb{R}^n$ be Lipschitz, where W is a nonempty open subset of $\mathbb{R}^n$, and let $V: W \to \mathbb{R}$ be a Lyapunov function for a fixed point $\bar{x} \in W$. Then $\bar{x}$ is stable. If V is a strict Lyapunov function, then $\bar{x}$ is asymptotically stable.

Proof. Let $\mathcal{U}$ be an open set with $\bar{x} \in \mathcal{U} \subset W$, and let $\rho > 0$ be such that the closure of the ball $B_\rho = \{x \in \mathbb{R}^n : |x - \bar{x}| < \rho\}$ is contained in $\mathcal{U}$. Set $m = \min\{V(x) : |x - \bar{x}| = \rho\} > 0$ and $\mathcal{U}' = \{x \in B_\rho : V(x) < m\}$. $\mathcal{U}'$ is an open set s.th. $\bar{x} \in \mathcal{U}' \subset \mathcal{U}$. Given $x \in \mathcal{U}'$, the E&U theorem guarantees an interval $(-a, a) \subset \mathbb{R}$ s.th. $\phi_t(x)$ exists in $\mathcal{U}'$ for all $t \in (-a, a)$. Let $(-a, b)$ with $b > 0$ be an interval of existence of $\phi_t(x)$ in $\mathcal{U}'$. Since $\overline{\mathcal{U}'} \subset W$, the lemma provides an extension of $\phi_t(x)$ to an interval $(-a, c)$, with $c > b$, in W. Since V is a Lyapunov function, $\frac{d}{dt} V(\phi_t(x)) = \nabla V \cdot f|_{\phi_t(x)} \le 0$, so $V(\phi_t(x)) < m \ \forall \ t \in [0, c)$. Since $\phi_t(x)$ is continuous in t, it cannot cross ∂B_ρ, where $V(x)$ attains a minimum of m. Thus $\phi_t(x) \in \mathcal{U}' \ \forall \ t \in [0, c)$. We conclude that $\phi_t(x)$ exists in $\mathcal{U}' \subset \mathcal{U}$ for all $t \in (-a, \infty)$. This proves stability.

To prove asymptotic stability when V is strict, let us assume that ∇V is continuous (otherwise a more technical proof follows the lines of the proof of the lemma). Let $x \in \mathcal{U}'$ (as above) be given. Since $V(\phi_t(x))$ is decreasing, $\exists \ V_0$ s.th. $V(\phi_t(x)) \searrow V_0$ as $t \to \infty$. If $V_0 > 0$, the continuity of V and $V(\bar{x}) = 0$ provide $\delta > 0$ such that $V(x) < V_0 \ \forall \ x \in B_\delta = \{x \in \mathbb{R}^n : |x - \bar{x}| < \delta\}$. Since $\nabla V \cdot f$ is continuous and strictly negative on $\bar{B}_\rho \setminus B_\delta$, it has a maximal value there, $-\mu = \max\{\nabla V \cdot f|_x : x \in \bar{B}_\rho \setminus B_\delta\} < 0$. $\forall \ t \ge 0$, since $\phi_t(x) \in \bar{B}_\rho \setminus B_\delta$, $\frac{d}{dt} V(\phi_t(x)) = \nabla V \cdot f|_{\phi_t(x)} \le -\mu$. But then for $t > m\mu^{-1}$, $V(\phi_t(x)) = V(x) + \int_0^t \nabla V \cdot f|_{\phi_s(x)} \, ds \le m - t\mu < 0$, which contradicts the positivity of V in $W \setminus \{\bar{x}\}$. We conclude that $V_0 = 0$. Let $\delta > 0$ be given with $\bar{B}_\delta \subset B_\rho$. Since V is continuous and nonzero in $W \setminus \{\bar{x}\}$, $\min\{V(x) : x \in \bar{B}_\rho \setminus B_\delta\} > 0$, and since $\phi_t(x) \in B_\rho$ $\forall\, t \geq 0$ and $V(\phi_t(x)) \to 0$ as $t \to \infty$, we conclude that $\exists\, t_0 \in \mathbb{R}$ s.th. $t > t_0 \Rightarrow \phi_t(x) \in B_\delta$. This proves convergence of $\phi_t(x)$ to $\bar{x}$ and hence the asymptotic stability of $\bar{x}$. ∎

Example (GH p. 5) Damped nonlinear oscillator $m\ddot{x} + \alpha\dot{x} + k(x + x^3) = 0$

In 1^{st}-order form:

$$\left.\begin{aligned} \dot{x} &= y \\ \dot{y} &= -m^{-1}\big(k(x + x^3) + \alpha y\big) \end{aligned}\right\} =: f(x,y)$$

Set

$$V(x,y) = \underbrace{\frac{m}{2}y^2 + k\left(\frac{x^2}{2} + \frac{x^4}{4}\right)}_{E(x,y) = \text{"energy"}} + \beta\left(xy + \frac{\alpha x^2}{2}\right)$$

One can show that, if $\beta > 0$ is small enough, then there is a nbhd N about the f.p. $(0,0)$ such that $V(x,y) > 0$ and $\nabla V \cdot f < 0$ in $N \setminus \{\bar{0}\}$. Thus the origin is asymptotically stable.

The Hartman – Grobman theorem says that linearization about a fixed point is locally faithful if no eigenvalues have vanishing real part.

GH p. 13 Theorem (Hartman-Grobman) Let $\bar{x}$ be a fixed point of the ODE $\dot{x} = f(x)$, with all eigenvalues of DF having non-zero real part. Then there exists an open set $\mathcal{U}$ containing $\bar{x}$ and an open set W of $\mathbb{R}^n$ containing 0 and a homeomorphism $h: \mathcal{U} \to W$ such that $h(\bar{x}) = 0$ and $\forall\, x \in \mathcal{U}$

$$e^{t\,Df(\bar{x})} hx = h(\phi_t(x)),$$

for all $t \in \mathbb{R}$ for which $\phi_t(x) \in \mathcal{U}$.

Let $\dot{x} = Ax$ be a real homogeneous linear constant-coefficient ODE such that all eigenvalues of A have non-zero real parts. The stable manifold of the fixed point 0 is just the span E^s of generalized eigenspaces corresponding to the eigenvalues with negative real part, i.e., the real invariant space corresponding to the Jordan blocks for those eigenvalues. Similarly, the unstable manifold is the corresponding space E^u for the eigenvalues with positive real part.

GH p. 13 Theorem (stable-manifold theorem) Let $f(\bar{x}) = 0$ and let all eigenvalues of $Df(\bar{x})$ have nonzero real part. Then E^s and E^u for the linearization $\dot{y} = Df(\bar{x})\,y$ are the tangent spaces to $W^s(\bar{x})$ and $W^u(\bar{x})$; and these manifolds are as smooth as f is.