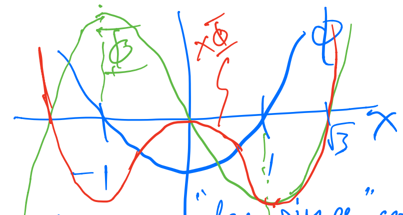
Van der Pol's oscillator

$$\underbrace{\ddot{x} + \alpha \phi(x)\dot{x}}_{\text{exact derivative} = \dot{y}} + x = \beta p(t)$$



$\phi(x)$: "damping" coeff.
negative $\Rightarrow$ gain
positive $\Rightarrow$ loss

with $y = \dot{x} + \alpha \Phi(x)$ $\quad [\Phi' = \phi,\ \Phi(0) = 0]$

$$\Rightarrow \begin{cases} \dot{x} = y - \alpha \Phi(x) \\ \dot{y} = -x + \beta p(t) \end{cases}$$

$p(t)$: oscillatory external forcing

Take $\phi(x) = x^2 - 1$
$\Phi(x) = \frac{x^3}{3} - x$

$\xi(t) = -x(t)$, $\eta(t) = -y(t)$

$$\Rightarrow \begin{cases} \dot{\xi} = \eta - \alpha \Phi(\xi) \\ \dot{\eta} = -\xi - \beta p(t) \end{cases}$$

Thus the system is invariant under rotation by 180^0 in the plane.

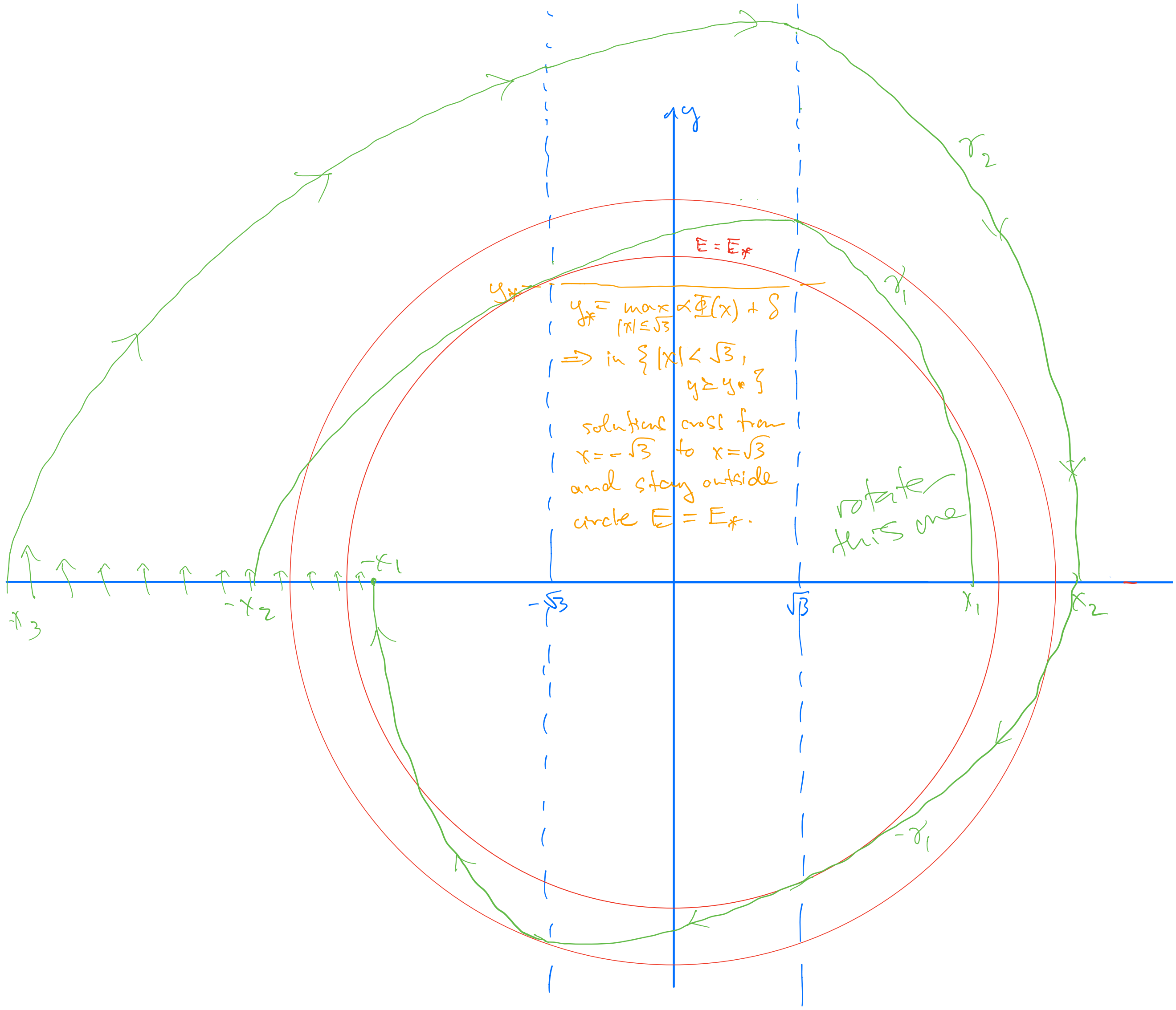
Set $\quad E = \frac{1}{2}(x^2 + y^2)$

$$\dot{E} = -\alpha x \Phi(x) + \beta y p(t), \quad \text{So } \dot{E} > 0 \text{ for } |x| < \sqrt{3}$$

Unforced system: $\beta = 0$

This is an autonomous system in the plane.
Let's first prove that it has a periodic orbit using P-B theorem.

We create a trapping region as follows:
In the region $\{|x| < \sqrt{3},\ y > \max_{|x|<\sqrt{3}} \alpha\Phi(x) + \delta\}$, $\dot{E} > 0$ and $\dot{x} > \delta$,
so solutions do cross over from $x = -\sqrt{3}$ to $x = \sqrt{3}$ ~ see γ_1 in this region below.
γ_1 continues backward in time until it hits $(-x_2, 0)$ and forward until it hits $(x_1, 0)$, with $x_1 < x_2$. A similar trajectory goes from $(-x_3, 0)$ to $(x_2, 0)$ with $x_2 < x_3$. The curve $\gamma_2 \cup -\gamma_1$ is a trajectory by symmetry. The region bounded by γ_2, γ_1 and $[-x_3, -x_1]$ is a trapping region.



Weak loss/gain : $0 < \alpha \ll 1$

The solution of the simple harmonic oscillator ($\alpha = 0$) is

$$\gamma_t(x,y) = (x,y) \begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix} = r\left(\cos(\varphi_0 - t), \sin(\varphi_0 - t)\right)$$

Take $\gamma_{-t}(x,y)$ to be a rotating coordinate system for any value of α. In the new coordinates

$$(u,v) = \gamma_{-t}(x,y) = (x,y)\begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix}$$

we obtain

$$\dot{u} = -\alpha\, \Phi(u\cos t - v\sin t)\cos t = -\alpha \cos t\, \Phi(r\cos(\varphi + t))$$

$$\dot{v} = \alpha\, \Phi(u\cos t - v\sin t)\sin t = \alpha \sin t\, \Phi(r\cos(\varphi + t))$$

Heuristic: If α is small, pretend that (u,v) is approximately constant over a period of 2π and integrate in t over a period.

$$\int_{-\pi}^{\pi} \Phi(r\cos(\varphi+t)) \cos t \, dt = \left[\Phi(r\cos(\cdot)) * \cos\right](-\varphi)$$

$$-\dot{u} + i\dot{v} = \alpha \int_{-\pi}^{\pi} \Phi(r\cos(\varphi+t))\, e^{it}\, dt = \alpha\, e^{-i\varphi} \int_{-\pi}^{\pi} \Phi(r\cos t)\, e^{it}\, dt$$

$$= \alpha\, e^{-i\varphi} \int_{-\pi}^{\pi} \Phi(r\cos t) \cos t \, dt = \alpha\, G(r)\, e^{-i\varphi}$$

$$\Rightarrow \begin{cases} \dot{u} = -\alpha\, G(r)\cos\varphi + \dots = \alpha\, u\, H(r) + \dots \\ \dot{v} = -\alpha\, G(r)\sin\varphi + \dots = \alpha\, v\, H(r) + \dots \end{cases}, \quad H(r) = -\frac{G(r)}{r}$$

$$\Rightarrow \begin{cases} \dot{r} = \alpha\, r\, H(r) + O(\alpha^2) \\ \dot{\theta} = 0 + O(\alpha^2) \end{cases}$$

From later theorem

This indicates that the periodic trajectory emanates from the circular trajectories for the harmonic oscillator at radii equal to the roots of $H(r)$.

For $\Phi(x) = \frac{x^3}{3} - x$, $\quad H(r) = \frac{1}{2}\left(1 - \frac{r^2}{4}\right)$, so the periodic orbit emanates from the circular one at $r=2$ for the harmonic oscillator.

For large α, we use the time variable αt.

Weak loss/gain : $\alpha \gg 1$

$$\begin{cases} x' = \frac{y}{\alpha} - \Phi(x) \\ y' = -\frac{x}{\alpha} \end{cases} \qquad \left.\begin{aligned} z' &= \frac{dz}{d(\alpha t)} \\ \hat{y} &= x' + \Phi(x) \\ &= \frac{y}{\alpha} \end{aligned}\right\} \Rightarrow \begin{cases} x' = \hat{y} - \Phi(x) \\ \hat{y}' = -\frac{x}{\alpha^2} \end{cases}$$