Moduli Interpretations for Noncongruence Modular Curves

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Theorem (Lubotsky-Segal, 2003)

$$\lim_{n \to \infty} \frac{\#\{\textit{noncongruence subgroups of index } n\}}{\#\{\textit{congruence subgroups of index } n\}} = \infty$$

Examples

Recall the classical congruence subgroups

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\Gamma(N) &:= \{\gamma \in \mathrm{SL}_2(\mathbb{Z}) : \gamma \equiv \left[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right] \mod N\} \\
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3. Γ is congruence if G is abelian.

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Solution: Allow for ramification at ∞. I.e., consider covers of punctured elliptic curves $E - \infty$.

Why? Because $\pi_1(E - \infty) \cong F_2$ (free group of rank 2) which has plenty of nonabelian quotients!

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$$1 \longrightarrow \pi_1^{\mathbb{L}}(E_s^\circ, g(s)) \longrightarrow \pi_1'(E^\circ, g(s)) \underset{g_*}{\overset{f_*}{\rightleftarrows}} \pi_1(S, s) \longrightarrow 1$$

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The construction of $\pi_1^{\mathbb{L}}(E^\circ/S, g, s)$ is independent of g, s (up to inner automorphisms), and commutes with arbitrary base change.

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If $S = \mathrm{Spec}\ k$ for an algebraically closed field k, then

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In general

$$\mathcal{H}om_S^{\text{sur-ext}}(\pi_1(E^\circ/S), G)(S) \quad \subset \quad \mathrm{Hom}^{\text{sur}}(F_2, G)/\mathrm{Inn}(G)$$

Suppose E°/S admits a section $g : S \to E^\circ$, Suppose E°/S admits a section $g : S \to E^\circ$, then for any covering space $X^\circ \to E^\circ$, we may consider g^*X°.

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Theorem

There is a canonical bijection

$$\mathcal{H}om_S^{\text{sur-ext}}(\pi_1(E^\circ/S), G)(S) \quad \sim \quad \{\text{Connected principal } G\text{-bundles } X^\circ/E^\circ \text{ s.t. } g^*X^\circ \text{ is completely decomposed}\}/\cong$$

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2. A morphism $h : (E'/S', \alpha') \to (E/S, \alpha)$ is a fiber-product diagram

$$\begin{array}{ccc} E' & \longrightarrow & E \\ \downarrow & & \downarrow \\ S' & \longrightarrow & S \end{array}$$

such that $h^*(\alpha) = \alpha'$

"Forgetting" the level structure α yields a morphism (ie., functor)

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The "forget level structure" morphism $p : \mathcal{M}_G \to \mathcal{M}_{1,1}$ is finite etale, and for any $E_0/\overline{\mathbb{Q}}$,

$$p^{-1}(E_0/\overline{\mathbb{Q}}) = \mathcal{H}om_{\overline{\mathbb{Q}}}^{sur\text{-}ext}(\pi_1(E_0^\circ/\overline{\mathbb{Q}}), G)(\overline{\mathbb{Q}}) \cong Hom^{sur}(F_2, G)/Inn(G)$$

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There's a classical exact sequence

$$1 \longrightarrow \mathrm{Inn}(F_2) \longrightarrow \mathrm{Aut}(F_2) \longrightarrow \mathrm{GL}_2(\mathbb{Z}) \longrightarrow 1$$

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Theorem

The monodromy action of $\pi_1((\mathcal{M}_{1,1})_{\overline{\mathbb{Q}}}) \cong \widehat{SL_2(\mathbb{Z})}$ on $p^{-1}(E_0/\overline{\mathbb{Q}})$ is via outer automorphisms of F_2.

Main Results

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Let $\varphi : F_2 \twoheadrightarrow G$ be a surjective homomorphism, then we may think of $[\varphi] \in p^{-1}(E_0/\overline{\mathbb{Q}})$, and let $\Gamma_{[\varphi]} := \mathrm{Stab}_{\mathrm{SL}_2(\mathbb{Z})}([\varphi])$.

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5. *If G is abelian, then $\Gamma_{[\varphi]}$ is congruence.*

Example: $G = \mathbb{Z}/N\mathbb{Z}$

There is one $\mathrm{SL}_2(\mathbb{Z})$-orbit on

$$\mathrm{Hom}^{\mathsf{sur}}(F_2, \mathbb{Z}/N\mathbb{Z})/\mathrm{Inn}(\mathbb{Z}/N\mathbb{Z}) = \mathrm{Hom}^{\mathsf{sur}}(\mathbb{Z}^2, \mathbb{Z}/N\mathbb{Z})$$

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$$\varphi : \left[\begin{smallmatrix} m \\ n \end{smallmatrix}\right] \mapsto n \mod N$$

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$$\mathrm{Hom}^{\mathrm{sur}}(\mathbb{Z}^2, (\mathbb{Z}/N\mathbb{Z})^2)$$

where each orbit corresponds to a possible determinant, and the stabilizers are all $\Gamma(N)$.

Example: $G = A_5$

There are three $\mathrm{SL}_2(\mathbb{Z})$-orbits on $\mathrm{Hom}^{\mathsf{sur}}(F_2, A_5)/\mathrm{Inn}(A_5)$, with reps

$$\varphi_1 : \begin{array}{ccc} x & \mapsto & (23)(45) \\ y & \mapsto & (152) \end{array} \qquad \varphi_2 : \begin{array}{ccc} x & \mapsto & (23)(45) \\ y & \mapsto & (142) \end{array} \qquad \varphi_3 : \begin{array}{ccc} x & \mapsto & (23)(45) \\ y & \mapsto & (14352) \end{array}$$

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Ie, Γ is noncongruence iff $e < d$ ($p_\ell(\Gamma)$ is large in $\mathrm{SL}_2(\mathbb{Z}/\ell)$).

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(ie, they're pairwise coprime) and $\{\left[\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right], \left[\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right]\}$ generate $\mathrm{SL}_2(\mathbb{Z})$.

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Thus, $p_\ell(\Gamma_{[\varphi]}) = \mathrm{SL}_2(\mathbb{Z}/\ell)$, so $e = 1 < d$, hence $\Gamma_{[\varphi]}$ is noncongruence.

Theorem

If $G = S_n$ $(n \geq 4)$, A_n $(n \geq 5)$, or $PSL_2(\mathbb{F}_p)$ $(p \geq 5)$, then there exists a surjection $F_2 \to G$ such that $\Gamma_{[\varphi]}$ is noncongruence.

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Conjecture

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1. *For every nonabelian finite simple group G, every surjection $\varphi : F_2 \to G$ has $\Gamma_{[\varphi]}$ noncongruence.*
2. *For every finite group G, either all surjections $\varphi : F_2 \to G$ have $\Gamma_{[\varphi]}$ congruence, or all surjections have $\Gamma_{[\varphi]}$ noncongruence.*

Which subgroups of $\mathrm{SL}_2(\mathbb{Z})$ appear as $\Gamma_{[\varphi]}$?

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Theorem (Asada, 2001)

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Corollary

Every modular curve is the quotient of some fine moduli scheme $M_{[\varphi]} = \mathcal{H}/\Gamma_{[\varphi]}$ for some homomorphism $\varphi : F_2 \to G$.

If we replace the sheaf $\mathcal{H}om_S^{\mathsf{sur}}(\pi_1(E^\circ/S), G)$ with its quotient

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Theorem (Ellenberg-McReynolds, 2011)

Every finite index subgroup of $\Gamma(2)$ *containing* $\pm I$ *is a Veech group.*