Computing modularity of some Calabi-Yau threefolds

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Preliminaries

Borcea studied crepant resolutions of quotients of the form

$$(E_1 \times E_2 \times E_3)/\langle \iota \times \iota \times \mathrm{id}, \iota \times \mathrm{id} \times \iota \rangle$$

He showed the Calabi-Yau threefolds of CM-type in this family were the varieties with each of the E_i having CM.

Calabi-Yau Threefolds

Let $E_k/\mathbb{C}$ be an elliptic curve with an automorphism of order $k = 3, 4$ or 6, denoted ι_k.

Consider the triple product $E_k \times E_k \times E_k$ with an action by

$$G_k = \langle \iota_k \times \iota_k^{k-1} \times \mathrm{id}, \iota_k \times \mathrm{id} \times \iota_k^{k-1} \rangle.$$

For any subgroup H_k of G_k we may consider the quotient

$$(E_k \times E_k \times E_k)/H_k,$$

and a crepant resolution.

Interest

We have two main focuses with the Calabi-Yau threefolds of this form.

(1) Study the 'easy' to understand rigid Calabi-Yau threefolds in this construction,

(2) Push this towards the non-rigid threefolds and see how much can be extended and said.

Modularity of X_4

Start with $k = 4$ and G_4, e.g.,

- $E_4 : y^2 = x^3 - x$,
- $\iota_4(x, y) = (-x, iy)$, and
- $X_4 := \widetilde{E_4^3 / G_4}$.

- This is defined over $\mathbb{Q}$.
- This is a rigid threefold. ($h^{1,1} = 90$.)
- Thus, it is modular by Gouvêa-Yui (Serre).

L-function of X_4

Note that

$$H^3_\ell(\widetilde{\overline{E_4}^3/G_4}) \simeq H^3_\ell(\overline{E_4}^3/G_4),$$

and

$$H^3_\ell(\overline{E_4}^3)^{G_4} = (H^1_\ell(\overline{E_4}) \otimes H^1_\ell(\overline{E_4}) \otimes H^1_\ell(\overline{E_4}))^{G_4},$$

which is 2-dimensional.

To make things explicit, we will work with the Galois representation

$$\rho : G_{\mathbb{Q}} \to \mathrm{Aut}_{\overline{\mathbb{Q}_\ell}}((V_\ell(E_4) \otimes V_\ell(E_4) \otimes V_\ell(E_4))^{G_4}),$$

where $V_\ell(E_4) := T_\ell(E_4)^\vee \otimes \overline{\mathbb{Q}_\ell}$.

Galois representation on X_4

For simplicity, denote the endomorphism ι_4 of E_4 by $[i]$ and the induced action on $V_\ell(E_4)$ by $[i]_*$. Note that $[i]^2 = [-1]$.

For any $\sigma \in G_\mathbb{Q}$ and $(x, y) \in E_4(\overline{\mathbb{Q}})$

$$\begin{aligned}\sigma([i](x,y)) &= (-\sigma(x), \sigma(i)\sigma(y)) \\ &= [\sigma(i)]\sigma((x,y)) = \chi(\sigma)[i]\sigma((x,y)),\end{aligned}$$

where $\chi : G_\mathbb{Q} \to \overline{\mathbb{Q}_\ell}^\times$ is the (non-trivial) Dirichlet character of $\mathbb{Q}(i)$. So

$$\begin{aligned}i\sigma_*(v) &= \sigma_*(iv) = \sigma_*([i]_*(v)) = (\sigma \circ [i])_*(v) \\ &= ([\chi(\sigma)i] \circ \sigma])_*(v) = [\chi(\sigma)]_*([i]_*(\sigma_*))(v) \\ &= \chi(\sigma)[i]_*(\sigma_*(v)).\end{aligned}$$

Of particular interest, if c denotes complex conjugation, then $w = c_*(v)$ is in the $(-i)$-eigenspace of $[i]_*$.

Frobenius on X_4

For a prime $p \neq 2$, if $\chi(\mathrm{Frob}_p) = 1$ the above shows the action of $(\mathrm{Frob}_p)_*$ is of the form

$$\begin{pmatrix} \alpha_p & 0 \\ 0 & \beta_p \end{pmatrix}$$

and otherwise when $\chi(\mathrm{Frob}_p) = -1$, of the form

$$\begin{pmatrix} 0 & h_p \\ k_p & 0 \end{pmatrix}$$

Thus, on the threefold we have the respective actions

$$\begin{pmatrix} \alpha_p^3 & 0 \\ 0 & \beta_p^3 \end{pmatrix}, \quad \begin{pmatrix} 0 & h_p^3 \\ k_p^3 & 0 \end{pmatrix}.$$

Trace of Frobenius

Thus we find

$$\mathrm{tr}(\rho(\mathrm{Frob}_p)) = \alpha_p^3 + \beta_p^3 = (\alpha_p + \beta_p)^3 - 3p(\alpha_p + \beta_p).$$

Coincidentally,

Lemma

Let ψ be a Hecke character of an imaginary quadratic field K and suppose f_ψ, the cusp form associated to ψ, has trivial Nebentypus. Suppose that we have Fourier q-expansions

$$f_\psi = \sum a_n q^n \qquad f_{\psi^3} = \sum b_n q^n.$$

Then

$$b_p = a_p^3 - 3pa_p.$$

L-function of X_4

Theorem

Let H be a subgroup of G_4 such that X_4, a crepant resolution of E_4^3/H, is a rigid Calabi-Yau threefold. We have

$$L(X_4, s) = L(s, \chi_4^3)$$

where χ_4 is the Hecke character such that

$$L(E_4, s) = L(s, \chi_4).$$

Twists

Let $E_4(n)$ denote the twist $y^2 = x^3 - n^2x$, and $X_4(n)$ a crepant resolution of $E_4(n)^3/G_4$. One similarly gets

Theorem

Let H be a subgroup of G_4 such that a crepant resolution $X_4(n)$ of $E_4(n)^3/H$ is a rigid Calabi-Yau threefold. We have

$$L(X_4(n), s) = L(s, \chi_4^3)$$

where χ_4 is the Hecke character such that

$$L(E_4(n), s) = L(s, \chi_4).$$

Similar for E_3/E_6 and twists.

Special values

Using work of Waldspurger, one may compute the vanishing and non-varnishing of the respective L-series found above.

Theorem

Let $E_4 : y^2 = x^3 - x$*, and* $X_4(-n)$ *a crepant resolution of* $E_4(-n)^3/G_4$*. For any odd square-free* $n \in \mathbb{N}$ *we have*

$$L(X_4(-n), 2) = \begin{cases} \frac{a_n^2}{\alpha\sqrt{n^3}} & \text{if } n \equiv 1 \pmod 8, \\ \frac{b_n^2}{\beta\sqrt{n^3}} & \text{if } n \equiv 3 \pmod 8, \\ 0 & \text{if } n \equiv 5, 7 \pmod 8. \end{cases}$$

where $\alpha, \beta \in \mathbb{C}^\times$ *and*

$$\sum a_n q^n = q - 3q^9 - 4q^{17} + \ldots \in S_{5/2}(128, \chi_{triv}),$$

$$\sum b_n q^n = -q^3 + 5q^{11} - 7q^{19} + \ldots \in S_{5/2}(128, \chi_{triv}).$$

Non-rigid construction

Suppose instead, we choose

$$H_4 = \langle \iota_4 \times \iota_4 \times \iota_4^2 \rangle \subset G_4,$$

and the quotient E_4^3/H_4.

We have $h^{2,1} = 1$ and so $Y_4 = \widetilde{E_4^3/H_4}$ is a *non-rigid* Calabi-Yau threefold.

What can we say now?

Galois representation

We still need only understand the Galois representation on $V_\ell(E_4)$, as

$$H^3_\ell(Y_4) \simeq (V_\ell(E_4)^{\otimes 3})^{H_4}$$

and as generators we can take

$$x \otimes x \otimes x, x \otimes x \otimes y,$$
$$y \otimes y \otimes x, y \otimes y \otimes y,$$

where (x, y) is a basis for $V_\ell(E_4)$.

Action of Frobenius

For $p \neq 2$, and $\chi : G_{\mathbb{Q}} \to \overline{\mathbb{Q}_\ell}$ as before, if $\chi(\mathrm{Frob}_p) = 1$ the action of Frobenius is given by

$$\begin{pmatrix} \alpha_p^3 & & & \\ & \alpha_p^2\beta_p & & \\ & & \alpha_p\beta_p^2 & \\ & & & \beta_p^3 \end{pmatrix}$$

and otherwise, if $\chi(\mathrm{Frob}_p) = -1$ the action is given by

$$\begin{pmatrix} & & & h_p^3 \\ & & h_p^2 k_p & \\ & h_p k_p^2 & & \\ k_p^3 & & & \end{pmatrix}$$

Trace of Frobenius

We have

$$\mathrm{tr}(\rho(\mathrm{Frob}_p)) = \begin{cases} \alpha_p^3 + \alpha_p^2\beta_p + \alpha_p\beta_p^2 + \beta_p^3 & \text{if } \chi(\mathrm{Frob}_p) = 1, \\ 0 & \text{otherwise} \end{cases}$$

Note, if $\chi(\mathrm{Frob}_p) = -1$, then $\alpha_p, \beta_p = \pm i\sqrt{p}$, so

$$\begin{aligned} \mathrm{tr}(\rho(\mathrm{Frob}_p)) &= \alpha_p^3 + \alpha_p^2\beta_p + \alpha_p\beta_p^2 + \beta_p^3 \\ &= \alpha_p^3 + \beta_p^3 + p(\alpha_p + \beta_p). \end{aligned}$$

Hence

$$L(Y_4, s) = L(\mathrm{Sym}^3 f_4, s) = L(\chi_4^3, s)L(\chi_4, s-1)$$

where χ_4 is the Hecke character associated to E_4.

Families

Consider again the triple product

$$E_4 \times E_4 \times E_4$$

with the action by $\iota_4 \times \iota_4 \times \iota_4^2$.

Replace this with

$$E_4 \times E_4 \times E$$

and the action by $\iota_4 \times \iota_4 \times \iota$, where E is any non-CM elliptic curve with hyperelliptic involution ι.

Galois representation

The Galois module of interest is now

$$\Big(V_\ell(E_4) \otimes V_\ell(E_4) \otimes V_\ell(E)\Big)^{\langle \iota_4 \times \iota_4 \times \iota \rangle},$$

so let $V_\ell(E)$ have basis (u, w) and (for a good prime p) eigenvalues γ_p, δ_p of Frob_p.

A basis for the Galois module is then given by

$$x \otimes x \otimes u, x \otimes x \otimes w,$$
$$y \otimes y \otimes u, y \otimes y \otimes w.$$

Trace of Frobenius

The CM-elliptic curve still gives rise to two cases. If $\chi(\mathrm{Frob}_p) = 1$ the action of Frobenius is given by

$$\begin{pmatrix} \alpha_p^2\gamma_p & & & \\ & \alpha_p^2\delta_p & & \\ & & \beta_p^2\gamma_p & \\ & & & \beta_p^2\delta_p \end{pmatrix}$$

and otherwise, if $\chi(\mathrm{Frob}_p) = -1$ the action is given by

$$\begin{pmatrix} & & h_p^2\gamma_p & \\ & & & h_p^2\delta_p \\ k_p^2\gamma_p & & & \\ & k_p^2\delta_p & & \end{pmatrix}$$

L-functions

Hence

$$\mathrm{tr}(\rho(\mathrm{Frob}_p)) = \begin{cases} (\alpha_p^2 + \beta_p^2)(\gamma_p + \delta_p) & \text{if } \chi(\mathrm{Frob}_p) = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Again, as $\alpha_p, \beta_p = \pm i\sqrt{p}$ when $\chi(\mathrm{Frob}_p) = -1$ we may simplify to

$$\mathrm{tr}(\rho(\mathrm{Frob}_p)) = (\gamma_p + \delta_p)(\alpha_p^2 + \alpha_p\beta_p + \beta_p^2 - p\chi(\mathrm{Frob}_p)).$$

Hence, the L-function is the product

$$L(f_E \times \mathrm{Sym}^2 f_4, s)L(f_E \otimes \chi, s-1)^{-1}.$$

Automorphy

$$L(f_E \times \operatorname{Sym}^2 f_4, s) L(f_E \otimes \chi, s-1)^{-1}.$$

The following shows the L-function is automorphic:

- $\operatorname{Sym}^2 f_4$ is automorphic on GL_3 (Gelbart-Jaquet),
- $f_E \times \operatorname{Sym}^2 f_4$ is automorphic on GL_6 (Kim-Shahidi),
- The product of automorphic L-functions is automorphic (Langlands).

Construction with E_6

Similarly with E_3/E_6.

Except there are no non-rigid Calabi-Yau threefolds Y of the form

$$E_6 \times E_6 \times E_6/H$$

with $h^{2,1}(Y) = 1$ *coming from the Künneth component*.

What else can we do?

More non-rigid

Are there any subgroups of H of G_k such that

$$(E_k \times E_k \times E_k)/H$$

only has *nice* singularities? Yes! For example, on

$E_4 \times E_4 \times E_4$, take

$$H = \langle \iota_4^2 \times \iota_4^2 \times \mathrm{id}, \iota_4^2 \times \mathrm{id} \times \iota_4^2 \rangle.$$

Borcea construction

Letting α_p, β_p be the eigenvalues of Frob_p ($p \neq 2$) on E_4, if $\chi(\mathrm{Frob}_p) = 1$, the action of Frobenius on E_4^3/H is given by

$$\begin{pmatrix} \alpha_p^3 &&&&&&& \\ & \alpha_p^2\beta_p &&&&&& \\ && \alpha_p^2\beta_p &&&&& \\ &&& \alpha_p^2\beta_p &&&& \\ &&&& \alpha_p\beta_p^2 &&& \\ &&&&& \alpha_p\beta_p^2 && \\ &&&&&& \alpha_p\beta_p^2 & \\ &&&&&&& \beta_p^3 \end{pmatrix}$$

If $\chi(\mathrm{Frob}_p) = -1$...

Trace of Frobenius

Hence, we have

$$\mathrm{tr}(\rho(\mathrm{Frob}_p)) = \begin{cases} \alpha_p^3 + 3p(\alpha_p + \beta_p) + \beta_p^3 & \text{if } \chi(\mathrm{Frob}_p) = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Again, the CM condition gives

$$\mathrm{tr}(\rho(\mathrm{Frob}_p)) = \alpha_p^3 + 3p(\alpha_p + \beta_p) + \beta_p^3$$

and so the L-function is (unsurprisingly?)

$$L(\chi_4^3, s)L(\chi_4, s-1)^3.$$

This time, a similar statement holds for E_3/E_6.

Further questions and remarks

- Is there a relationship between the special values of E_k and X_k?
- What is the rank of $\mathrm{CH}^2(X_k)_0$?
- Do similar (rigid) threefolds exist for each of the weight 4 CM newforms defined over $\mathbb{Q}$?
- Can we use this approach with the Borcea-Voisin construction?

Thank you

Thank you to the attendees for coming!

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