# The method of brackets

## A heuristic method for integration

Conference in honor of Robert Perlis

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# Collaborators

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Karen Kohl, University of Southern Alabama

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Lin Jiu, RISC, Johannes Kepler University, Austria

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Given a function

$$f : [a, b] \to \mathbb{R}$$

determine

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# A quartic analog

**Theorem**

*For $m \in \mathbb{N}$ and $a > -1$*

$$\int_0^\infty \frac{dx}{(x^4 + 2ax^2 + 1)^{m+1}} = \frac{\pi}{2} \frac{P_m(a)}{[2(a+1)]^{m+1/2}}$$

$$P_m(a) = \sum_{l=0}^{m} d_{l,m} a^l$$

$$d_{l,m} = 2^{-2m} \sum_{k=l}^{m} 2^k \binom{2m-2k}{m-k} \binom{m+k}{m} \binom{k}{l}$$

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Coefficients $d_{l,m}$ have many interesting properties

# Where else do the appear?

$$N_{0,4}(a;m) = \int_0^\infty \frac{dx}{(x^4 + 2ax^2 + 1)^{m+1}}$$

$$\sqrt{a + \sqrt{1+c}} = \sqrt{a+1} + \frac{1}{\pi\sqrt{2}} \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} N_{0,4}(a, k-1) c^k$$

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# Double square roots

In view of the relation of the quartic integral with double square roots, I looked for integrals having this function in the integrand

The famous table by Gradshteyn and Ryzhik has

$$\int_0^\infty \frac{dx}{(1+x^2)^{3/2}\left[\phi(x)+\sqrt{\phi(x)}\right]^{1/2}} = \frac{\pi}{2\sqrt{6}}$$

as entry 3.248.5 $\phi(x)$ is a simple rational function
Beautiful

Unfortunately it is incorrect

I have no idea what the correct value is.

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# Using integrals to define functions

$$\log x = \int_1^x t^{-1}\, dt$$

You have to make sure you have created a new function
Here is a famous one

$$\Gamma(x) = \int_0^\infty e^{-t} t^{x-1}\, dt$$

You get interesting functions integrating simple ones

And you also get interesting numbers

$$\gamma = -\Gamma'(1)$$

Then you can combine them.

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# $L_n = \int_0^1 \log^n \Gamma(q)\, dq$

**Theorem**

*(L. Euler)*

$$L_1 := \int_0^1 \log \Gamma(q)\, dq = \log \sqrt{2\pi}$$

**Theorem**

*(O. Espinosa, V.M., 2002)*

$$\begin{aligned} L_2 &:= \int_0^1 \log^2 \Gamma(q)\, dq \\ &= \frac{\gamma^2}{12} + \frac{\pi^2}{48} + \frac{1}{3}\gamma L_1 + \frac{4}{3}L_1^2 - (\gamma + 2L_1)\frac{\zeta'(2)}{\pi^2} + \frac{\zeta''(2)}{2\pi^2} \end{aligned}$$

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# Problem

We asked in 2002: what is

$$L_3 = \int_0^1 \log^3 \Gamma(q)\, dq \ \ ?$$

# Borwein-Bailey-Crandall finally got it in 2015

## Theorem

$$\begin{aligned}
L_3 &= \frac{3}{4}\left(\frac{\zeta(3)}{\pi^2} + \frac{L_1}{3}\right) A^2 \\
&- \frac{3}{2}\left(\frac{\zeta'(2,1)}{\pi^2} + 2L_1\frac{\zeta'(2)}{\pi^2}\right) A + \frac{3}{2} L_1 \frac{\zeta''(2)}{\pi^2} \\
&+ L_1^3 + \frac{\pi^2}{16} L_1 + \frac{3}{16}\zeta(3) \\
&- \frac{3}{8\pi^2}\left(\omega_{1,1,0}(1,1,1) - 2\omega_{1,0,1}(1,1,1)\right)
\end{aligned}$$

$$\omega(r,s,t) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{1}{n^r m^s (n+m)^t}$$

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$$\omega(r,s,t) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{1}{n^r m^s (n+m)^t}$$

Tornhein-Witten-zeta functions

$$\omega(r,s,t) = \frac{1}{\Gamma(t)} \int_0^1 \mathrm{Li}_r(\sigma)\mathrm{Li}_s(\sigma)(-\log\sigma)^{t-1}\, dt$$

$$\mathrm{Li}_s(x) = \sum_{n=1}^{\infty} \frac{x^n}{n^s} \text{ for } |x| < 1 \text{ and } \mathrm{Re} s > 1$$

$$\omega_{1,0,0}(r,s,t) = \frac{\partial}{\partial r}\omega(r,s,t)$$

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# A problem

Invent a new function

$$\zeta(s) = \frac{1}{\Gamma(s)} \int_0^\infty \frac{x^{s-1}\,dx}{e^x - 1}$$

Riemann did it.

`Evaluate the integral`

$$\int_0^\infty \frac{(1-12t^2)}{(1+4t^2)^3} \int_{1/2}^\infty \log|\zeta(\sigma+it)|\,d\sigma\,dt$$

$$\int_0^\infty \frac{(1-12t^2)}{(1+4t^2)^3} \int_{1/2}^\infty \log|\zeta(\sigma+it)|\,d\sigma\,dt = \frac{\pi(3-\gamma)}{32}$$

This might be hard

Is equivalent to the Riemann hypothesis.

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# The method of brackets

Start with the expansion

$$f(x) = \sum_{n=0}^{\infty} a_n x^n$$

Integrate term by term

$$\int_0^{\infty} f(x)\, dx = \sum_{n=0}^{\infty} a_n \int_0^{\infty} x^n\, dx$$

Now make sense of this

This is the goal of the rest of the talk

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$$\langle a \rangle := \int_0^\infty x^{a-1}\,dx \quad \text{for } a \in \mathbb{R}$$

this is the *bracket* corresponding to $a \in \mathbb{R}$.
Given the expansion

$$f(x) = \sum_{n=0}^{\infty} c_n x^{\alpha n + \beta - 1} \qquad c_n,\ \alpha,\ \beta \in \mathbb{C}$$

integrating term by term gives the value of the integral

$$\int_0^\infty f(x)\,dx = \sum_n c_n \langle \alpha n + \beta \rangle$$

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The integrals are divergent. Regularize.

We need to decide rules of evaluation for *bracket series*

We need to be able to produce bracket series in an efficient manner.

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# Rules to generate bracket series

**Rule 1**:

$$(a_1 + a_2 + \cdots + a_r)^{\alpha} \to$$

$$\sum_{n_1, n_2, \cdots, n_r} \phi_{1,2,\cdots,r} a_1^{n_1} \cdots a_r^{n_r} \frac{\langle -\alpha + n_1 + n_2 + \cdots + n_r \rangle}{\Gamma(-\alpha)}$$

$$\phi_n = \frac{(-1)^n}{\Gamma(n+1)}$$

$$\phi_{1,2} = \phi_{n_1} \phi_{n_2}$$

# Rules for brackets. Continuation

**Rule 2. Evaluation of a bracket series**:

$$\sum_n \phi_n f(n) \langle \alpha n + \beta \rangle \to \frac{1}{\alpha} f(n^*) \Gamma(-n^*)$$

where $n^*$ solves the equation $\alpha n + \beta = 0$.

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# Rules for brackets. Continuation

**Rule 3**:

$$\sum_{n_1,n_2} \phi_{n_1,n_2} f(n_1,n_2)\langle a_{11}n_1 + a_{12}n_2 + c_1\rangle\langle a_{21}n_1 + a_{22}n_2 + c_2\rangle$$

$$\rightarrow \frac{1}{|a_{11}a_{22} - a_{12}a_{21}|} f(n_1^*, n_2^*)\Gamma(-n_1^*)\Gamma(-n_2^*)$$

where $(n_1^*, n_2^*)$ solves the linear system obtained by the vanishing of brackets

We do not assign a value if the determinant vanishes.

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# Wallis' formula

$$J_m := \int_0^\infty \frac{dx}{(x^2+1)^{m+1}} = \int_0^\infty (1+x^2)^{-m-1}\,dx$$

$$(1+x^2)^{-m-1} \mapsto \sum_{n_1,n_2} \phi_{1,2} \frac{\langle m+1+n_1+n_2\rangle}{\Gamma(m+1)} 1^{n_1} x^{2n_2}$$

$$J_m \mapsto \sum_{n_1,n_2} \phi_{1,2} \frac{1}{\Gamma(m+1)} \langle m+1+n_1+n_2\rangle \langle 2n_2+1\rangle$$

System: $m+1+n_1+n_2=0, \quad 2n_2+1=0$

Solution: $n_1^* = -(m+1/2), \quad n_2^* = -1/2.$

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# Wallis' formula. Continuation

Rule 3 implies

$$J_m = \frac{1}{2}\frac{\Gamma(-n_1^*)\,\Gamma(-n_2^*)}{\Gamma(m+1)} = \frac{\Gamma(m+1/2)\,\Gamma(1/2)}{2\Gamma(m)}$$

The values

$$\Gamma(m) = m! \text{ and } \Gamma(m+1/2) = \frac{\sqrt{\pi}}{2^{2m}}\frac{(2m)!}{m!}$$

produce

$$\int_0^\infty \frac{dx}{(x^2+1)^{m+1}} = \frac{\pi}{2}\frac{\binom{2m}{m}}{2^{2m}}$$

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# A Bessel integral

$$I := \int_0^\infty J_0(ax)\sin(bx)dx$$

$$J_0(ax) = \sum_{m=0}^{\infty} \phi_m \frac{a^{2m}}{\Gamma(m+1)2^{2m}} x^{2m}$$

$$\sin(bx) = \sum_{n=0}^{\infty} \phi_n \frac{\Gamma(n+1)}{\Gamma(2n+2)} b^{2n+1} x^{2n+1}$$

$$I = \frac{\sqrt{\pi}}{2} \sum_{m,n} \phi_{m,n} \frac{a^{2m} b^{2n+1}}{2^{2m+2n}\Gamma(m+1)\Gamma(n+3/2)} \langle 2m+2n+2 \rangle.$$

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Choose $m$ as the free parameter: $n^* = -m-1$.

$$I = \frac{\sqrt{\pi}}{b} \sum_{m} \phi_m \frac{1}{\Gamma\left(-m+\frac{1}{2}\right)} \left(\frac{a}{b}\right)^{2m} = \frac{1}{\sqrt{b^2-a^2}} \quad |a| < |b|.$$

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# A multi-dimensional example: 4.638.3 in GR

$$I_n(s) = \int_0^\infty \int_0^\infty \cdots \int_0^\infty \frac{x_1^{p_1-1} x_2^{p_2-1} \cdots x_n^{p_n-1} \; dx_1 dx_2 \cdots dx_n}{(1 + (r_1x_1)^{q_1} + \cdots + (r_nx_n)^{q_n})^s},$$

$$\text{denominator} \mapsto \sum_{k_0,k_1,\cdots,k_n} \phi_{0,\cdots,n} \prod_{j=1}^{n} (r_jx_j)^{q_jk_j} \frac{\langle s + k_0 + \cdots + k_n \rangle}{\Gamma(s)}.$$

$$I_n \mapsto \sum_{k_0,k_1,\cdots,k_n} \phi_{0,\cdots,n} \prod_{j=1}^{n} (r_jx_j)^{q_jk_j} \frac{\langle s + k_0 + \cdots + k_n \rangle}{\Gamma(s)} \prod_{j=1}^{n} \langle p_j + q_jk_j \rangle.$$

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# A multi-dimensional example. Continuation

The solutions are $k_0 = -s + \sum_{j=1}^{n} \frac{p_j}{q_j}$ and $k_j = -\frac{p_j}{q_j}$ for $1 \leq j \leq n$.

$$I_n = \frac{1}{\Gamma(s)} \Gamma\left( s - \sum_{j=1}^{n} \frac{p_j}{q_j} \right) \prod_{j=1}^{n} \frac{\Gamma\left(\frac{p_j}{q_j}\right)}{q_j r_j^{p_j}}.$$

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# Ising integrals

$$C_{n,k} := \frac{4}{n!}\int_0^\infty \int_0^\infty \cdots \int_0^\infty \left(\sum_{j=1}^{n}(u_j + 1/u_j)^2\right)^{-k-1} \frac{du_1}{u_1}\frac{du_2}{u_2}\cdots\frac{du_n}{u_n}$$

reduces to (Jon Borwein-David Bradley)

$$C_{n,k} = \frac{2^{n-k+1}}{n!\,k!}\int_0^\infty t^k K_0^n(t)\,dt$$

where

$$K_0(x) = \lim_{\nu\to 0}\frac{\pi}{2}\frac{I_{-\nu}(x) - I_\nu(x)}{\sin \pi\nu}$$

is the modified Bessel function.

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# Ising integrals. Continuation.

The only known values

$$C_n := C_{n,1}$$

$$C_1 = 2, \qquad C_2 = 1,$$

$$C_3 = L_{-3}(2) = \sum_{n=0}^{\infty} \left( \frac{1}{(3n+1)^2} - \frac{1}{(3n+2)^2} \right),$$

$$C_4 = \frac{7}{12}\zeta(3)$$

These are the only ones we can evaluate with brackets.

There must be a reason

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These are the only ones we can evaluate with brackets.

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# Ising integrals. Continuation.

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# Difficulties with the method

$$I_2 = \frac{1}{2} C_{2,1} := \int_0^\infty \int_0^\infty \frac{dx\, dy}{xy(x + y + 1/x + 1/y)^2}$$

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4 indices and 3 brackets

One free variable

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$$\sum_{n_1,n_2,n_3,n_4} \phi_{1,2,3,4} \langle n_1 - n_3 \rangle \langle n_2 - n_4 \rangle \langle 2 + n_1 + n_2 + n_3 + n_4 \rangle$$

$n_1$ free: $n_2^* = -n_1 - 1$, $n_3^* = n_1$, $n_4^* = -n_1 - 1$

$$\begin{aligned} I_2 &= \sum_{n_1} \frac{1}{\det(A)} \phi_1 \Gamma(-n_2^*) \Gamma(-n_3^*) \Gamma(-n_4^*) \\ &= \frac{1}{2} \sum_{n_1} (-1)^{n_1} \Gamma(n_1 + 1) \Gamma(-n_1). \end{aligned}$$

Divergent

Discard it

The same occurs with the other three choices of free variable.

# Difficulties. Continuation

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Every form produces divergent integrals

The same occur with

$$I_2 = \int_0^\infty \int_0^\infty \frac{xy\,dx\,dy}{(xy(x+y)+x+y)^2}$$

$$\text{But} \quad I_2 = \int_0^\infty \int_0^\infty \frac{xy\,dx\,dy}{(xy(x+y)+(x+y))^2}$$

*Finally gives* $I_2 = \dfrac{1}{2}$

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# Integrals coming from Feynman diagrams

The figure shows interaction of three particles

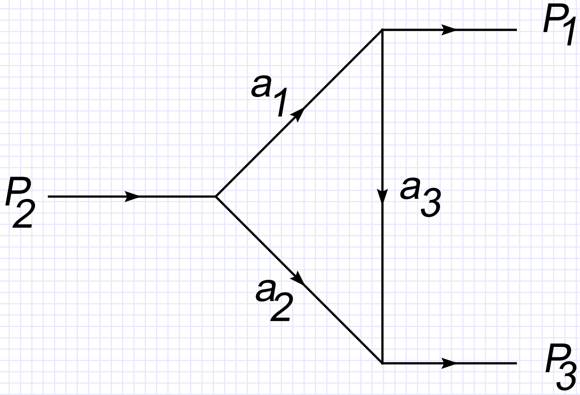


Figure : The triangle

# Integrals coming from Feynman diagrams. Continuation.

Schwinger parametrization gives

$$
\begin{aligned}
G &= \frac{(-1)^{-D/2}}{\Gamma(a_1)\Gamma(a_2)\Gamma(a_3)} \int_0^\infty \int_0^\infty \int_0^\infty \frac{x_1^{a_1-1} x_2^{a_2-1} x_3^{a_3-1}}{(x_1+x_2+x_3)^{D/2}} \\
&\times \exp(x_1 m_1^2 + x_2 m_2^2 + x_3 m_3^2) \\
&\times \exp\left(-\frac{C_{11} P_1^2 + 2C_{12} P_1 \cdot P_2 + C_{22} P_2^2}{x_1+x_2+x_3}\right) dx_1 dx_2 dx_3.
\end{aligned}
$$

The coefficients $C_{i,j}$ are given by

$$C_{11} = x_1(x_2+x_3), \; C_{12} = x_1 x_3, \; C_{22} = x_3(x_1+x_2).$$

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Conservation of momentum gives $P_3 = P_1 + P_2$ and then

$$
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&\times \exp\left(x_1 m_1^2 + x_2 m_2^2 + x_3 m_3^2\right) \\
&\times \frac{\exp\left(-\frac{x_1x_2P_1^2 + x_2x_3P_2^2 + x_3x_1P_3^2}{x_1+x_2+x_3}\right)}{(x_1+x_2+x_3)^{D/2}} dx_1 dx_2 dx_3 .
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# Integrals coming from Feynman diagrams. Continuation.

**Problem**. Evaluate $G = G(P_1, P_2, m_i, D, a_i)$.

Special case $m_1 = m_2 = m_3 = 0$ and $P_1^2 = P_2^2 = 0$.

$$\begin{aligned} G &= \frac{(-1)^{-D/2}}{\Gamma(a_1)\Gamma(a_2)\Gamma(a_3)} \\ &\times \int_{\mathbb{R}_+^3} x_1^{a_1-1} x_2^{a_2-1} x_3^{a_3-1} \frac{\exp\left(-\frac{x_1 x_3}{x_1+x_2+x_3} P_3^2\right)}{(x_1+x_2+x_3)^{D/2}} \, dx_1 \, dx_2 \, dx_3. \end{aligned}$$

The method of brackets gives

$$G = \frac{(-1)^{-D/2}}{\Gamma(a_1)\Gamma(a_2)\Gamma(a_3)} \sum_{n_1} \sum_{n_2} \sum_{n_3} \sum_{n_4} \phi_{1234} (P_3^2)^{n_1} \frac{\Delta_1 \Delta_2 \Delta_3 \Delta_4}{\Gamma(D/2+n_1)},$$

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# Integrals coming from Feynman diagrams. Continuation.

The brackets $\Delta_j$ are

$$\begin{aligned}\Delta_1 &= \langle D/2 + n_1 + n_2 + n_3 + n_4 \rangle,\\ \Delta_2 &= \langle a_1 + n_1 + n_2 \rangle,\\ \Delta_3 &= \langle a_2 + n_3 \rangle,\\ \Delta_4 &= \langle a_3 + n_1 + n_4 \rangle.\end{aligned}$$

This problem has no free indices.

$$n_1^* = \frac{D}{2} - a_1 - a_2 - a_3,\ n_2^* = -\frac{D}{2} + a_2 + a_3,\ n_3^* = -a_2,\ n_4^* = -\frac{D}{2} + a_1 + a_2.$$

# Integrals coming from Feynman diagrams. Continuation.

This gives

$$
\begin{aligned}
G &= \frac{(-1)^{-D/2}}{\Gamma(a_1)\Gamma(a_2)\Gamma(a_3)}(P_3^2)^{D/2-a_1-a_2-a_3} \times \\
&\times \frac{\Gamma(a_1+a_2+a_3-\frac{D}{2})\Gamma(\frac{D}{2}-a_2-a_3)\Gamma(a_2)\Gamma(\frac{D}{2})\Gamma(\frac{D}{2}-a_1-a_2)}{\Gamma(D-a_1-a_2-a_3)}.
\end{aligned}
$$

# How to use divergent series

Start with the `exponential integral`

$$\mathrm{Ei}(-x) = -\int_1^\infty \frac{e^{-xt}}{t}\,dt = -\sum_{n_1=0}^{\infty} \phi_{n_1} x^{n_1} \int_1^\infty t^{n_1-1}\,dt$$

$$\begin{aligned}\int_1^\infty t^{n_1-1}\,dt &= \int_0^\infty (y+1)^{n_1-1}\,dy \\ &= \sum_{n_2,n_3} \phi_{n_2 n_3} \frac{\langle -n_1+1+n_2+n_3\rangle\langle n_2+1\rangle}{\Gamma(-n_1+1)}\end{aligned}$$

# How to use divergent series. Continuation

This gives

$$\begin{aligned}\mathrm{Ei}(-x) &= -\sum_{n_1,n_2,n_3} \phi_{n_1 n_2 n_3} x^{n_1} \frac{\langle -n_1+1+n_2+n_3\rangle\langle n_2+1\rangle}{\Gamma(-n_1+1)} \\ &= \sum_{n=0}^{\infty} \phi_n \frac{x^n}{n}\end{aligned}$$

a divergent series

This is the bracket version of the expansion

$$\mathrm{Ei}(-x) = \gamma + \ln(x) + \sum_{n=1}^{\infty} \phi_n \frac{x^n}{n}$$

# How to use divergent series. Continuation

This gives

$$\begin{aligned} \mathrm{Ei}(-x) &= -\sum_{n_1,n_2,n_3} \phi_{n_1 n_2 n_3} x^{n_1} \frac{\langle -n_1+1+n_2+n_3\rangle \langle n_2+1\rangle}{\Gamma(-n_1+1)} \\ &= \sum_{n=0}^{\infty} \phi_n \frac{x^n}{n} \end{aligned}$$

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We compute the Mellin transform

$$\begin{aligned}\int_0^\infty x^{s-1}\mathrm{Ei}(-bx)\,dx &= \sum_n \phi_n \frac{b^n}{n}\int_0^\infty x^{\mu+n-1}\,dx \\ &= \sum_n \phi_n \frac{b^n}{n}\langle \mu+n\rangle \\ &= -\frac{b^{-\mu}}{\mu}\Gamma(\mu).\end{aligned}$$

Exercise for the audience

$$\int_0^\infty x^{\nu-1}e^{-\mu x}\mathrm{Ei}(-\beta x)\,dx = -\frac{\Gamma(\nu)}{\nu(\beta+\nu)^\nu}\,{}_2F_1\left(1,\ \nu;\ \nu+1;\ \frac{\mu}{\beta+\mu}\right)$$

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Thanks for your attention.