Integer sequences and their valuations

Southern Regional Number Theory Conference

Modular Curves, Modular Forms, and Hypergeometric Functions

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LSU

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General project

Given a sequence of integers $\{x_n\}$ and a prime p,

determine properties of the sequence $\{\nu_p(x_n)\}$

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Perhaps create a version of OEIS for valuations

p-adic valuations

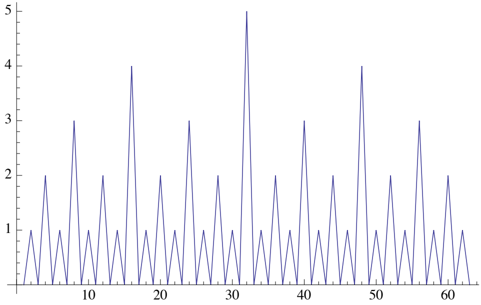
For a prime p and $n \in \mathbb{N}$ write

$$n = p^x \times b$$

with b not divisible by p.

The integer x is the p-**adic valuation of** n denoted by $x = \nu_p(n)$

The 2-adic valuation of n



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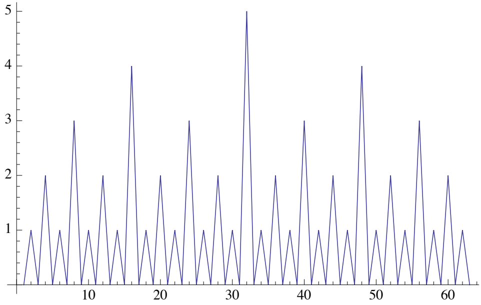
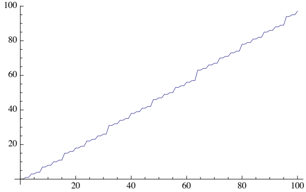
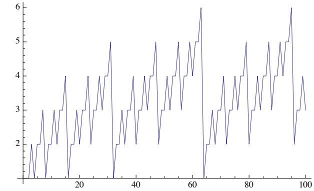


Is there an analytic formula for this expression?

The 2-adic valuation of $n!$ and deviation from asymptotics





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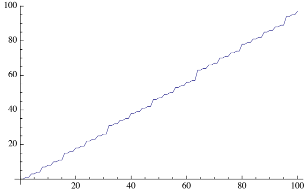

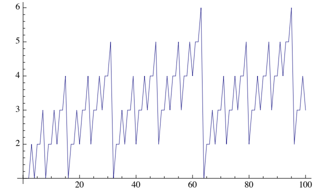

Is there an analytic formula for this expression?

The p-adic valuation of factorials

Theorem

Let $n \in \mathbb{N}$ and p prime. Then

$$\nu_p(n!) = \sum_{k=1}^{\infty} \left\lfloor \frac{n}{p^k} \right\rfloor$$

The p-adic valuation of factorials. An example.

Example

$$\nu_5(581!) = 143$$

$$\begin{aligned}
\nu_5(581!) &= \sum_{k=1}^{\infty} \left\lfloor \frac{581}{5^k} \right\rfloor \\
&= \left\lfloor \frac{581}{5} \right\rfloor + \left\lfloor \frac{581}{25} \right\rfloor + \left\lfloor \frac{581}{125} \right\rfloor \\
&= 116 + 23 + 4 \\
&= 143.
\end{aligned}$$

The number $581!$ ends with 143 zeros.

A theorem of Legendre.

Theorem

For $n \in \mathbb{N}$ *and* p *prime write*

$$n = a_0 + a_1 p + a_2 p^2 + \cdots + a_r p^r$$

the base p *representation of* n

Define $s_p(n) = a_0 + a_1 + \cdots + a_r$
then

$$\nu_p(n!) = \frac{n - s_p(n)}{p - 1}$$

A theorem of Legendre. An example.

Example

$581 = 1 \cdot 5^0 + 1 \cdot 5^1 + 3 \cdot 5^3 + 4 \cdot 5^3 = (4311)_5$

$$\nu_5(581!) = \frac{581 - (1+1+3+4)}{5-1} = \frac{572}{4} = 143$$

The valuation of n

Now write

$$n = \frac{n!}{(n-1)!}$$

to obtain

The valuation of n

Now write

$$n = \frac{n!}{(n-1)!}$$

to obtain

$$\nu_p(n) = \frac{-s_p(n) + 1 + s_p(n-1)}{p-1}$$

Sequences satisfying first order recurrences. Reducible case

Theorem

Assume $t_n = Q(n)t_{n-1}$
$Q \in \mathbb{Z}[x]$
$Q(x) = \prod_{j=1}^{m} (x - \beta_j) \times Q_1(x)$
$\beta_j \in \mathbb{Z}_p$
with $Q_1(x) \equiv 0 \bmod p$ *unsolvable.*
Then

$$\nu_p(t_n) = \frac{mn}{p-1} + O(\log n)$$

Sequences satisfying first order recurrences. Irreducible case

Theorem

Assume $t_n = Q(n) t_{n-1}$
$Q \in \mathbb{Z}[x]$ *irreducible over* $\mathbb{Z}_p$
$l =$ *Sup*$\{k : p^k | Q(i)$ *for some* $i \in \mathbb{Z}\}$
Then, with $m = \deg(Q)$ *and* $w = \lfloor l/m \rfloor$,

$$\nu_p(t_n) = \left(\sum_{k=1}^{w} \frac{m}{p^k} \right) \times n + (l - mw) \frac{n}{p^{w+1}} + O(1)$$

Central binomial coefficients

Central binomial coefficients

$$\binom{2n}{n} = \frac{(2n)!}{n!^2}$$

$$\nu_p\left(\binom{2n}{n}\right) = \frac{2s_p(n) - s_p(2n)}{p-1}$$

$$\nu_2\left(\binom{2n}{n}\right) = 2s_2(n) - s_2(2n) = s_2(n)$$

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Corollary

The central binomial coefficient is always even.
It is divisible exactly by 2 *if and only if* $n = 2^r$.

The 3-adic valuation of central binomial coefficients

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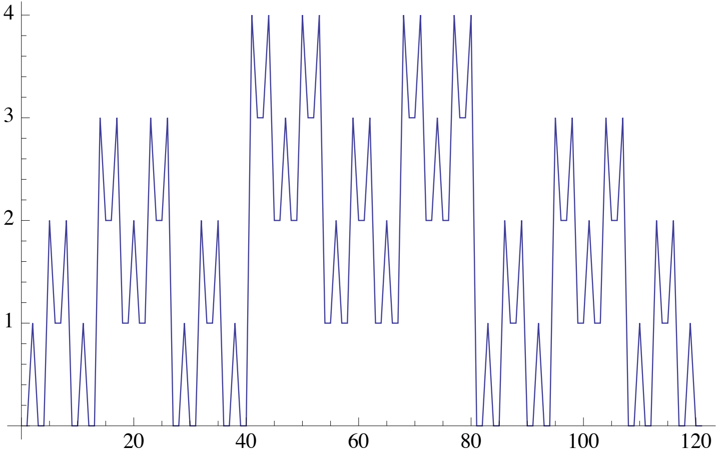

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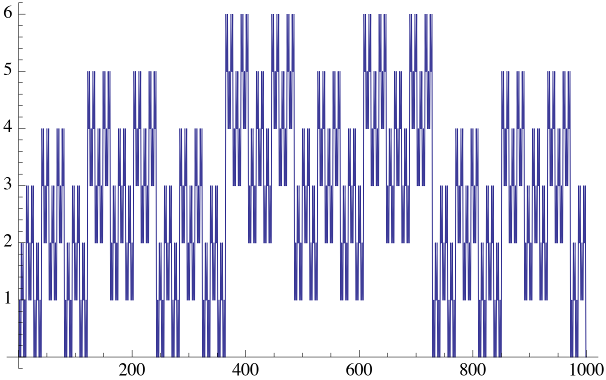

Data for $\nu_3\left(\binom{2n}{n}\right)$

$$0, 1, 0, 0, 2, 1, 1, 2, 0, 0, 1, 0, 0, 3, 2, 2, 3, 1, 1, 2, 1, 1, 3,$$
$$2, 2, 3, 0, 0, 1, 0, 0, 2, 1, 1, 2, 0, 0, 1, 0, 0, 4, 3, 3, 4, 2, 2,$$
$$3, 2, 2, 4, 3, 3, 4, 1, 1, 2, 1, 1, 3, 2, 2, 3, 1, 1, 2, 1, 1, 4, 3,$$
$$3, 4, 2, 2, 3, 2, 2, 4, 3, 3, 4, 0, 0, 1, 0, 0, 2, 1, 1, 2, 0, 0, 1,$$
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Compare with

$$0, 1, 0, 0, 1, 0, 0, 1, 0, 0, 1, 0, 0, 1, 0, 0, 1, 0, 0, 1, 0, 0, 1, 0$$

repeated with period 3

The first refinement

$\nu_3 \binom{2n}{n} - \{0, 1, 0\}$ (periodic extension)

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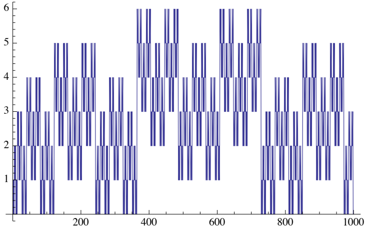
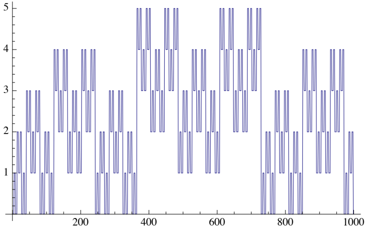
$\nu_3\binom{2n}{n} - \{0,\, 1,\, 0\}$ (periodic extension)





Further refinements

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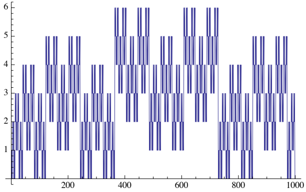

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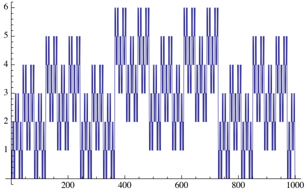

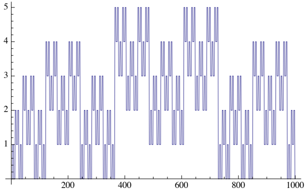

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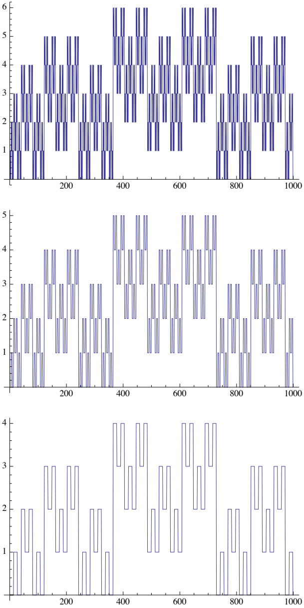


A simpler example: Back to the valuation of n

Observe that

$$n = \frac{n!}{(n-1)!}$$

Theorem

Let $n \in \mathbb{N}$ and p prime. Then

$$\nu_p(n) = \sum_{k=1}^{\infty} \left(\left\lfloor \frac{n}{p^k} \right\rfloor - \left\lfloor \frac{n-1}{p^k} \right\rfloor \right)$$

Each summand is periodic of period p^k.
Similar to Fourier series.

The ASM numbers

The numbers

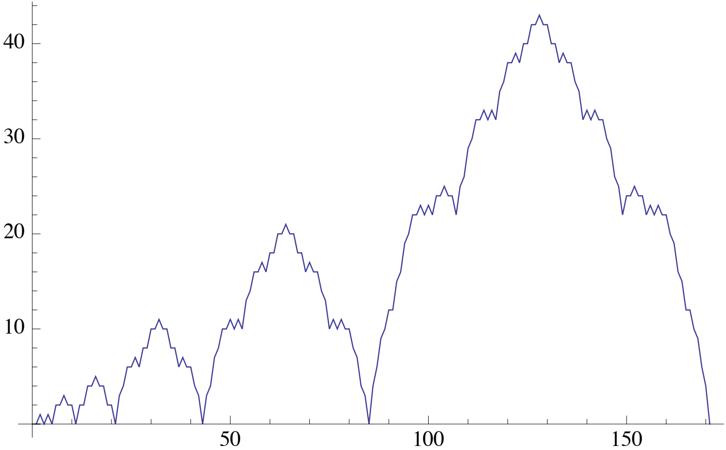
$$T_n = \prod_{j=0}^{n-1} \frac{(3j+1)!}{(n+j)!}$$

count the numbers of $n \times n$ matrices such that

- entries are $+1,\, 0,\, -1$
- non-zero entries alternate
- row sum and column sum is $+1$

Problem. Prove $T_n \in \mathbb{N}$ without counting.

The 2-adic valuation of ASM



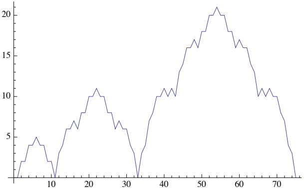
The Jacobstahl numbers

$$J_n = J_{n-1} + 2J_{n-2} \quad J_0 = J_1 = 1$$

Theorem

The ASM number T_n is odd if and only if n is a Jacobstahl number.

The 2-adic valuation of ASM numbers



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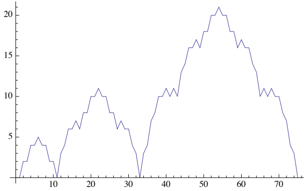


Theorem

(X. Sun - V.M.)

$$\nu_2(T(2^n)) = J_{n-1} \qquad \nu_2(T(J_n)) = 0$$

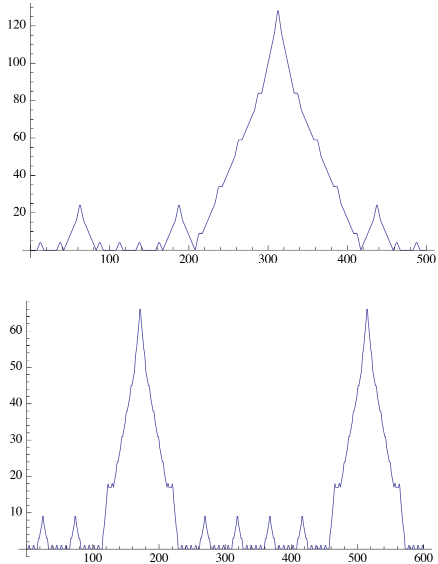
$$\textit{for } 0 < i \leq 2J_{n-3} \quad \nu_2(T(J_n + i)) = i + \nu_2(T(J_{n-2} + i))$$

$$\nu_2(T(j)) > 0 \textit{ for } J_n < j < 2^n - J_{n-2}$$

$$\nu_2(T(2^n - J_{n-2} + i)) = \nu_2(T(J_{n-1} + i)) + 2J_{n-3}$$

$$\nu_2(T(2^n - i)) = \nu_2(T(2^n + i))$$

The p-adic valuation of ASM numbers



The p-adic valuation of ASM numbers

Theorem

(E. Beyerstedt, X. Sun, V.M.)

$$\varphi_{j,p}(n) = \begin{cases} 0 & \text{if} \quad 0 \leq n \leq \left\lfloor \frac{p^j+1}{3} \right\rfloor \\ n - \left\lfloor \frac{p^j+1}{3} \right\rfloor & \text{if} \quad \left\lfloor \frac{p^j+1}{3} \right\rfloor + 1 \leq n \leq \frac{p^j-1}{2} \\ \left\lfloor \frac{2p^j+1}{3} \right\rfloor - n & \text{if} \quad \frac{p^j+1}{2} \leq n \leq \left\lfloor \frac{2p^j+1}{3} \right\rfloor \\ 0 & \text{if} \quad \left\lfloor \frac{2p^j+1}{3} \right\rfloor + 1 \leq n \leq p^j - 1. \end{cases}$$

Then

$$\nu_p(T_n) = \sum_{j=1}^{\infty} \varphi_{j,p}(n \bmod p^j)$$

each summand is of period p^j.

A generalization

The same phenomena should work for

$$T_n(q) = \prod_{j=0}^{n-1} \frac{(qj+1)!}{(n+j)!}$$

Question: what do these numbers count?

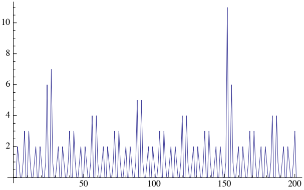
Stirling numbers of the second kind

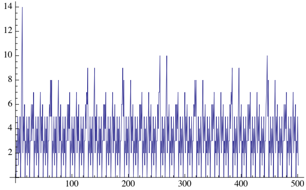
$$S(n,k) = S(n-1,k-1) + kS(n-1,k)$$

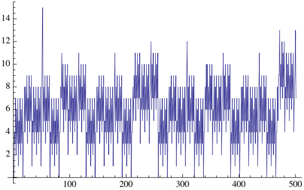
$$\sum_{k=1}^{\infty} S(n,k)x^n = \frac{1}{(1-x)(1-2x)(1-3x)\cdots(1-kx)}$$

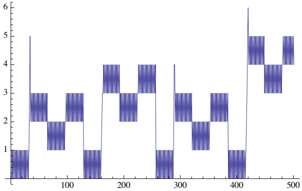
$$S(n,k) = \frac{1}{k!}\sum_{j=0}^{k-1}(-1)^j \binom{k}{j}(k-j)^n.$$

Valuations of Stirling numbers

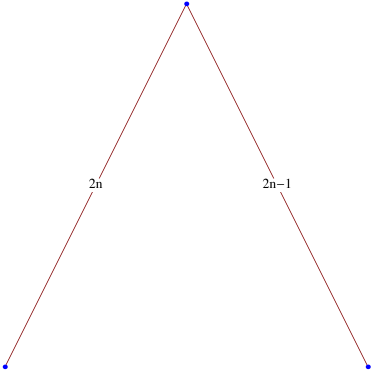








Valuation of Stirling numbers



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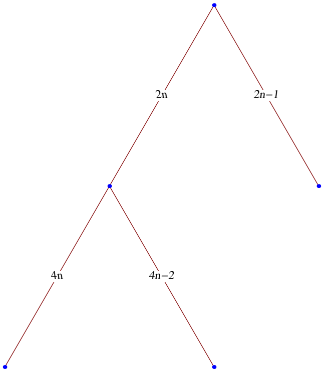

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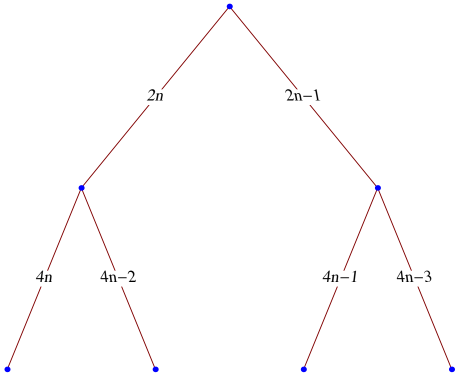

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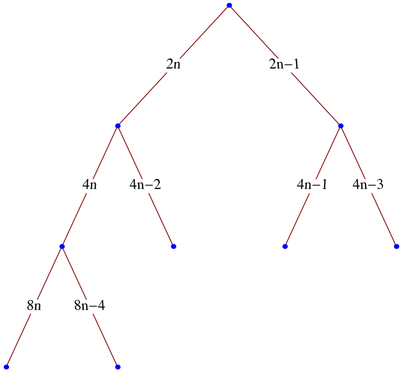

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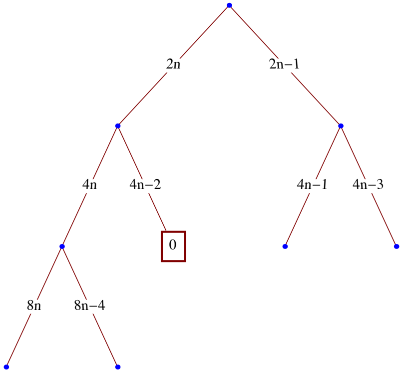

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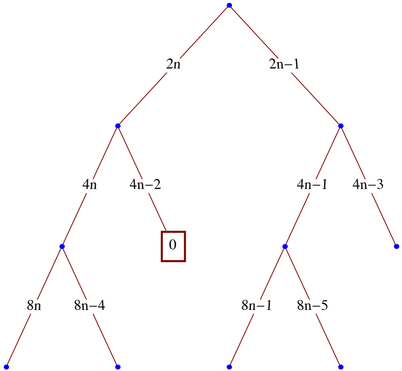

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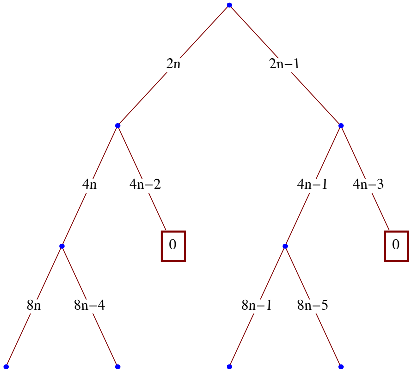

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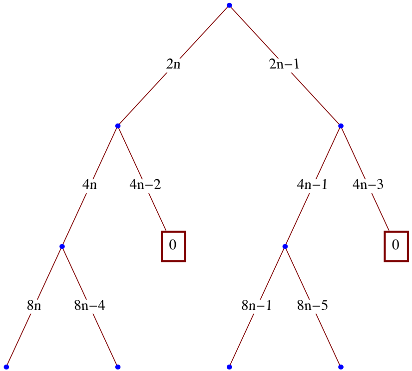

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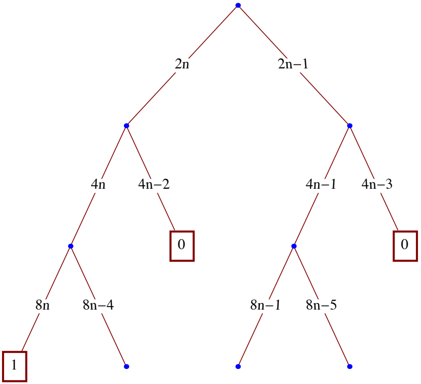

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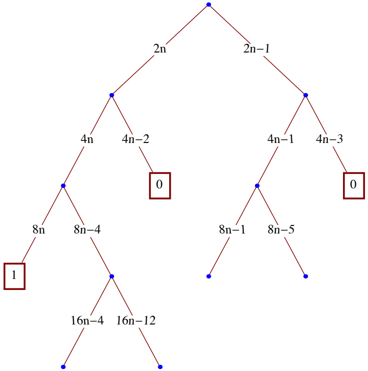

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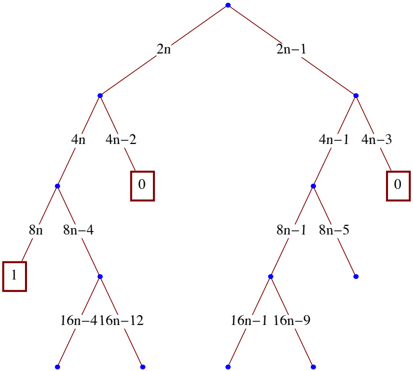

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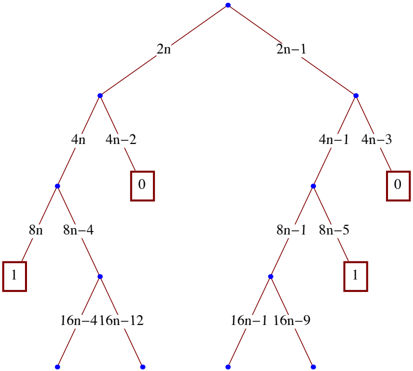

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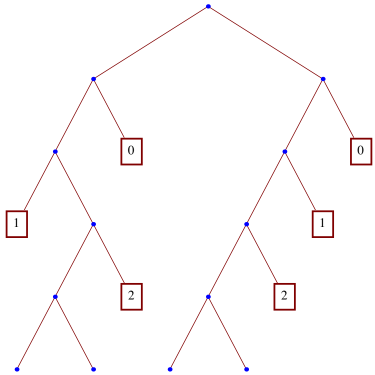

Valuation of Stirling numbers

Conjecture

Fix k, then the tree associated to $\nu_2(S(n,k))$ is a complete binary tree up a certain level. From that point on, each vertex has two descendents: one terminates and the other continues.

The Moll doctrine

Someone said

Every talk must contain a proof

The Moll doctrine

Someone said

Every talk must contain a proof

I say

Every talk must contain an integral

A quartic integral

Theorem

For $m \in \mathbb{N}$ *and* $a > -1$

$$\int_0^\infty \frac{dx}{(x^4 + 2ax^2 + 1)^{m+1}} = \frac{\pi}{2} \frac{P_m(a)}{[2(a+1)]^{m+1/2}}$$

$$P_m(a) = \sum_{l=0}^{m} d_{l,m} a^l$$

$$d_{l,m} = 2^{-2m} \sum_{k=l}^{m} 2^k \binom{2m-2k}{m-k} \binom{m+k}{m} \binom{k}{l}$$

Nice change of variables

Theorem

Given a function f, define

$$f_{\pm}(x) = f(x + \sqrt{x^2+1}) \pm f(x - \sqrt{x^2+1}).$$

Then

$$\int_{-\infty}^{\infty} f(x)\, dx = \int_{-\infty}^{\infty} f_{+}(y)\, dy + \int_{-\infty}^{\infty} f_{-}(y)\, \frac{y}{\sqrt{y^2+1}} dy.$$

Proof.

Let

$$y = \frac{x^2-1}{2x}$$

□

Nice change of variables. Continuation

The nice change of variables reduces the integral

$$\int_0^\infty \frac{dx}{(x^4 + 2ax^2 + 1)^{m+1}}$$

to a finite sum of integrals of the form

$$\int_0^\infty \frac{dx}{(x^2 + 1)^{j+1}}.$$

This was evaluated by J. Wallis before integration was invented.

The coefficients

$$d_{l,m} = 2^{-2m} \sum_{k=l}^{m} 2^k \binom{2m-2k}{m-k} \binom{m+k}{m} \binom{k}{l}$$

The original version was uglier

$$\begin{aligned} d_{l,m} &= \sum_{j=0}^{l} \sum_{s=0}^{m-l} \sum_{k=s+l}^{m} \frac{(-1)^{k-l-s}}{2^{3k}} \binom{2k}{k} \binom{2m+1}{2s+2j} \binom{m-s-j}{m-k} \\ &\times \binom{s+j}{j} \binom{k-s-j}{l-j} \end{aligned}$$

A short visit to Ramanujan's world

$$N_{0,4}(a,m) = \int_0^\infty \frac{dx}{(x^4 + 2ax^2 + 1)^{m+1}}$$

then

$$\sqrt{a + \sqrt{1+c}} = \sqrt{a+1} + \frac{1}{\pi\sqrt{2}} \sum_{k=1}^\infty \frac{(-1)^{k-1}}{k} N_{0,4}(a, k-1)\, c^k$$

Then apply Ramanujan Master Theorem to get $d_{l,m}$

Pretty identity

Two evaluations of the quartic integral produced

$$\sum_{k=0}^{m} 2^{-2k} \binom{2k}{k} \binom{2m-k}{m} = \sum_{k=0}^{m} 2^{-2k} \binom{2k}{k} \binom{2m+1}{2k}$$

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We have many proofs, but no combinatorial one.

Combinatorial aspects

Theorem

$d_{l,m}$ is unimodal (Boros, M.)

Theorem

$d_{l,m}$ is logconcave (Kauers, Paule)

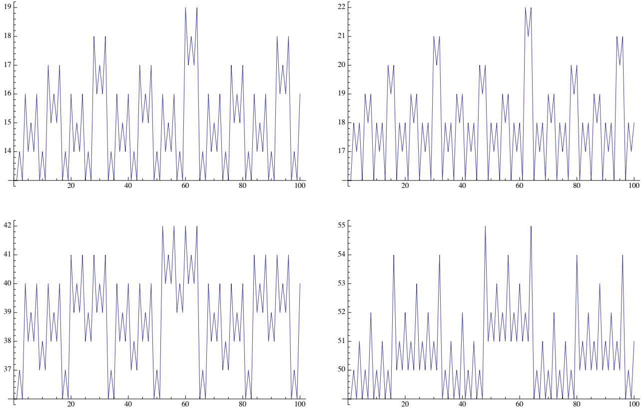
Theorem

$d_{l,m}$ is 2-logconcave (Chen et al.)

Conjecture

$d_{l,m}$ is infinitely logconcave

2-adic valuations of $d_{l,m}$



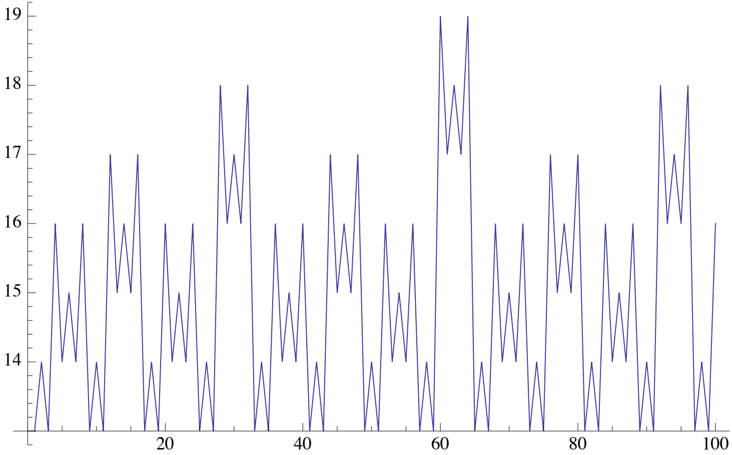
A binary tree associated to these valuations

Theorem

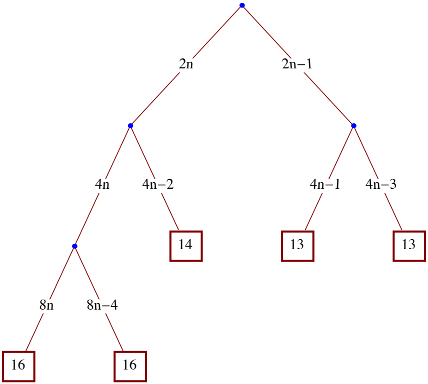
(X. Sun- V.M.) There is a binary tree that encodes the formulas for

$$\{\nu_2(d_{l,m}) : l \text{ is fixed and } m \geq l\}$$

$\nu_2(d_{5,m})$



The decision tree for $\nu_2(d_{5,m})$



Every tree is a formula

$$\nu_2(d_{5,2m}) = \nu_2(d_{5,2m+1})$$

$$\nu_2(d_{5,2m}) = \begin{cases} 13 + \nu_2\left(\frac{m+3}{4}\right) & \text{if } m \equiv 1 \bmod 4, \\ 13 + \nu_2\left(\frac{m+1}{4}\right) & \text{if } m \equiv 3 \bmod 4, \\ 14 + \nu_2\left(\frac{m+2}{4}\right) & \text{if } m \equiv 2 \bmod 4, \\ 16 + \nu_2\left(\frac{m}{8}\right) & \text{if } m \equiv 0 \bmod 8, \\ 16 + \nu_2\left(\frac{m+4}{8}\right) & \text{if } m \equiv 4 \bmod 8. \end{cases}$$

Problem

Extend this theory to odd primes.

Problems

- What properties of a sequence produces infinite trees?

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- Determine properties of the **Stirling random walks**

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- What properties of a sequence produces infinite trees?
- Determine properties of the **Stirling random walks**
- Do they extend to other recurrences?
- Develop orthogonality theory for valuation series.

All roads lead to Ramanujan

Define

$$a_n = \sum_{k=0}^{n} \frac{n!}{k!} n^k$$

All roads lead to Ramanujan

Define

$$a_n = \sum_{k=0}^{n} \frac{n!}{k!} n^k$$

Show

$$a_n = \frac{1}{2} n! e^n + (1-\theta) n^n$$

for some θ in the range between $\frac{1}{2}$ and $\frac{1}{3}$.

S. Ramanujan: Question 294
J. Indian Math. Soc. 1911

All roads lead to Ramanujan. Continuation

$$2, 10, 78, 824, 10970, 176112,$$

$$3309110, 71219584, 1727242866, 46602156800$$

Look for them in OEIS

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$$3309110, 71219584, 1727242866, 46602156800$$

Look for them in OEIS

This is sequence A063170.
They are called Schenker sums with n-th term

$$a_n = \int_0^\infty e^{-x}(x+n)^n \, dx$$

posted by M. Somos.

All roads lead to Ramanujan. Continuation

The asymptotics of Ramanujan's problem have been studied extensively

$$Q(n) = \int_0^\infty e^{-x} \left(1 + \frac{x}{n}\right)^{n-1} \, dx$$

$$a_n = (1 + Q(n)) n^n$$

See B. Berndt, Part II, Chapter 12, Entry 47.

G. McGarvey's conjecture

$$\nu_2(a_n) = \begin{cases} 1 & \text{if } n \text{ is odd} \\ n - s_2(n) & \text{if } n \text{ is even} \end{cases}$$

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$s_2(n)$ is the sum of binary digits of n.

Related to valuations by Legendre's formula

$$\nu_2(n!) = n - s_2(n)$$

The proof

Lemma

Assume $A(x)$ *polynomial with integer coefficients*

and every coefficient of A *is divisible by* r*, for some* $r \in \mathbb{Z}$*.*

Then the integer

$$\int_0^\infty A(x)e^{-x}\,dx$$

is divisible by r*.*

Corollary

If $A(x) \equiv B(x) \bmod r$ *then*

$$\int_0^\infty A(x)e^{-x}\,dx \equiv \int_0^\infty B(x)e^{-x}\,dx \bmod r.$$

The proof. Continuation

$$a_n = \int_0^\infty e^{-x} (x+n)^n \, dx$$

The proof. Continuation

$$a_n = \int_0^\infty e^{-x}(x+n)^n \, dx$$

$n = 2m+1$ odd

m even

$$a_n \equiv \int_0^\infty (1+x)e^{-x} \, dx = 2 \equiv 2 \bmod 4$$

m odd

$$a_n \equiv \int_0^\infty (3+x)e^{-x} \, dx = 78 \equiv 2 \bmod 4$$

Therefore $a_n \equiv 2 \bmod 4$ for all $n \in \mathbb{N}$.

The proof. Continuation

$n = 2m$ even

$$a_{2m} = \int_0^\infty (2m+x)^{2m} e^{-x}\, dx = \sum_{k=0}^{2m} \binom{2m}{k} (2m)^{2m-k} k!$$

$$t_{k,m} = \binom{2m}{k} (2m)^{2m-k} k!$$

satisfies $\quad 2m t_{2m-j+1,m} = j t_{2m-j,m}$

This gives the result in the form

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The last term in the sum controls the valuation.

The valuation for odd primes

Theorem

p *odd prime and* $n = mp$

$$\nu_p(a_n) = \nu_p(n!) = \frac{n - s_p(n)}{p-1}$$

residue 0 *case*

The valuation for odd primes

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Theorem

p odd prime and $n = mp + r$ with $0 < r < p$
(that is $n \equiv r \neq 0 \bmod p$)
then

$$a_n \equiv 0 \bmod p \text{ if and only if } a_r \equiv 0 \bmod p$$

modularity property

The valuation for odd primes. Some examples

Example

$p = 3$
$a_1 = 2$, $a_2 = 10$ not divisible by 3

$$\nu_3(a_n) = \begin{cases} \frac{1}{2}(n - s_3(n)) & \text{if } n \equiv 0 \bmod 3 \\ 0 & \text{if } n \not\equiv 0 \bmod 3 \end{cases}$$

Example

$p = 7$
$a_1 = 2$, $a_2 = 10$, $a_3 = 78$, $a_4 = 824$, $a_5 = 10970$, $a_6 = 176112$
not divisible by 7

$$\nu_7(a_n) = \begin{cases} \frac{1}{6}(n - s_7(n)) & \text{if } n \equiv 0 \bmod 7 \\ 0 & \text{if } n \not\equiv 0 \bmod 7 \end{cases}$$

The valuation for odd primes. The bad cases

Definition

p is called a Schenker prime if p divides a_r for some r in the range $1 \leq r \leq p-1$.

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$p = 5$ is a Schenker prime because it divides $a_2 = 10$.

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Theorem

Assume p is NOT a Schenker prime. Then

$$\nu_p(a_n) = \begin{cases} \frac{1}{p-1}(n - s_p(n)) & \textit{if } n \equiv 0 \bmod p \\ 0 & \textit{if } n \not\equiv 0 \bmod p \end{cases}$$

The case $p = 5$

$$\nu_5(a_{5n}) = \frac{n - s_5(n)}{4}$$

This is the residue 0 case.

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The case $\nu_5(a_{5n+2})$ requires further discussion.
The results depends on the digits of the expansion in base 5:

$$n = x_0 + x_1 \cdot 5 + x_2 \cdot 5^2 + x_3 \cdot 5^3 + \cdots$$

The Schenker primes

The list of Schenker primes begins with

$$5, 13, 23, 31, 37, 41, 43, 47, 53, 59, 61, 71, 79, 101, 103, 107, 109,$$

$$127, 137, 149, 157, 163, 173, 179, 181, 191, 197, 199, 211, 223.$$

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Smallest prime with a given index

index(p)	0	1	2	3	4	5
min(p)	2	3	41	179	1553	593

The next step in the project

$$R_n(a) := \sum_{k=0}^{n} \left(\frac{n!}{k!} \right)^a n^k$$

Valuations present interesting modular properties.

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Valuations present interesting modular properties.

Techniques developed for this example will lead to the analysis of partial sums of hypergeometric series.

As in the special case $p = 5$, experiments suggest that the tree is controlled by a small number of p-adic integers.

Fibonacci polynomials

$$F_n(x) = xF_{n-1}(x) + F_{n-2}$$

and

$$F_0(x) = 1, \; F_1(x) = x.$$

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Natural question:

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Natural question:
What can we say about $\nu_p(e_n)$?

Fibonacci polynomials. Continuation

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Theorem

The sequence $e_n \bmod m$ is periodic of period $2m$.

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Numbers e_j with $1 \leq j \leq 2p$ with $e_j \equiv 0 \bmod p$ are called roots.

Example

$e_{10} = 405845$, so e_{10} is a root modulo 11

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The other roots are e_{12} and e_{22}.

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For 343 out of the first 1000 primes, this is the only root.

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Strange fact: *the only prime up to* p_{50000} *with sum* $0 \bmod p$ is $p = 97$
Stranger fact: *if p is not prime, then 0 appears if*
$n = 2^m \times 97, \quad 0 \leq m \leq 4.$

The Yablonskii-Vorob'ev polynomials

This is a family of polynomials defined by

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$$P_{n+1}(x) = \frac{xP_n^2(x) - 4(P_n(x)P_n''(x) - (P_n'(x))^2}{P_{n-1}(x))}$$

with $P_0(x) = 1$ and $P_1(x) = x$.

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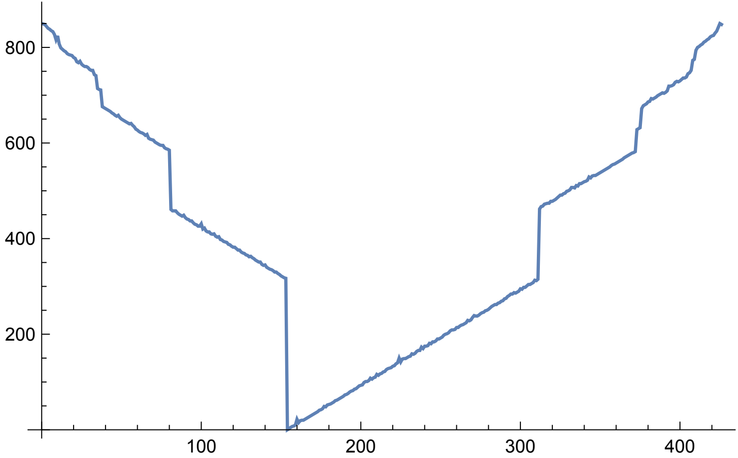
These are polynomials with integer coefiicients

Prime $p = 2$

Valuations of coefficients



Prime $p = 5$

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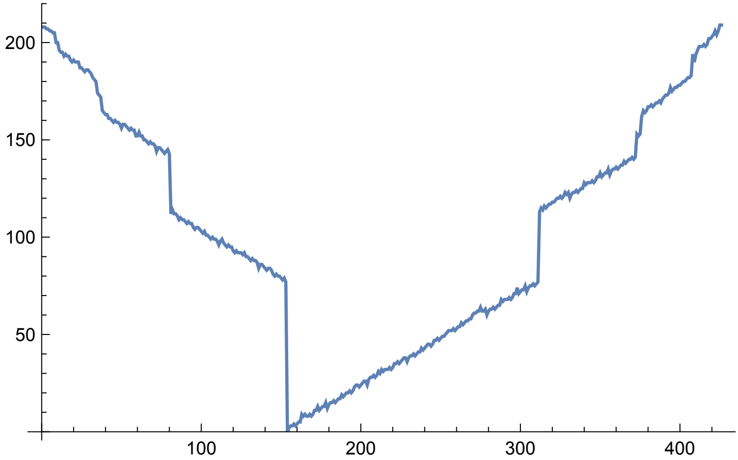
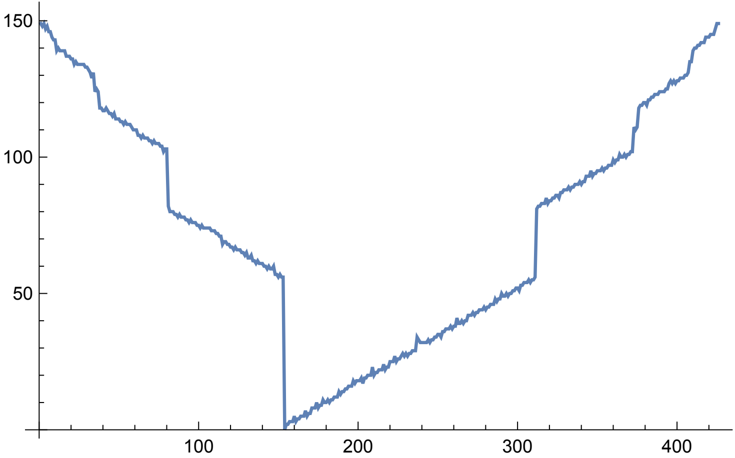


Prime $p = 7$

Valuations of coefficients



Prime $p = 3$

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Why?

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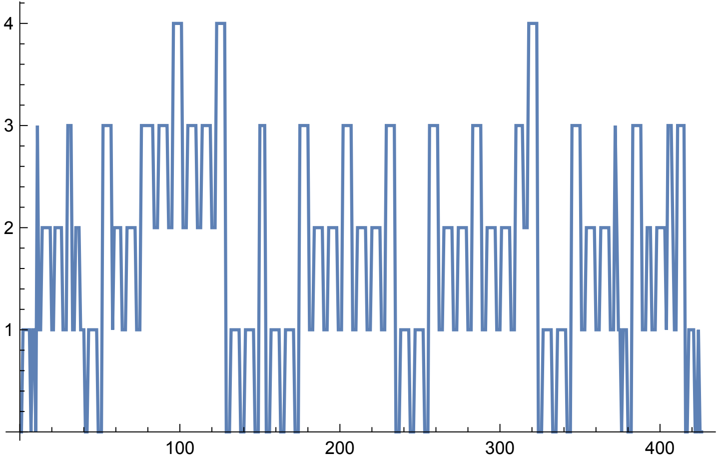

Subject of current work.

THANKS FOR YOUR ATTENTION