UNIFORM EXPONENT BOUNDS ON THE NUMBER OF PRIMITIVE EXTENSIONS OF NUMBER FIELDS

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Abstract. A folklore conjecture asserts the existence of a constant $c_n > 0$ such that $N_n(X) \sim c_n X$ as $X \to \infty$, where $N_n(X)$ is the number of degree n extensions $K/\mathbb{Q}$ with discriminant bounded by X. This conjecture is known if $n \leq 5$, but even the weaker conjecture that there exists an absolute constant $C \geq 1$ such that $N_n(X) \ll_n X^C$ remains unknown and apparently out of reach.

Here, we make progress on this weaker conjecture (which we term the "uniform exponent conjecture") in two ways. First, we reduce the general problem to that of studying relative extensions of number fields whose Galois group is an almost simple group in its smallest degree permutation representation. Second, for almost all such groups, we prove the strongest known upper bound on the number of such extensions. These bounds have the effect of resolving the uniform exponent conjecture for solvable groups, sporadic groups, exceptional groups, and classical groups of bounded rank. This is forthcoming work that grew out of conversations with M. Bhargava.