A VOLCANIC APPROACH TO CM POINTS ON SHIMURA CURVES

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ABSTRACT. A CM component of the ℓ-isogeny graph of elliptic curves over a finite field has a particular structure, that of an ℓ-volcano, away from certain CM orders. The structure of these "isogeny volcanoes" has seen much use in the study of CM elliptic curves over finite fields, originating with the 1996 PhD thesis of Kohel. Recent work of Clark and Clark–Saia leverages infinite depth versions of these graphs to study moduli of isogenies of CM elliptic curves over $\overline{\mathbb{Q}}$.

We will discuss an analogue of this work for abelian surfaces with quaternionic multiplication. A main result of our study is an algorithm for describing the Δ-CM locus on the Shimura curve $X_0^D(N)_{/\mathbb{Q}}$, for $\gcd(D, N) = 1$ and Δ any imaginary quadratic discriminant. Specifically, our work allows for an enumeration of all points in this locus with a given residue field and ramification index, generalizing a result from Jordan's 1981 thesis in the $N = 1$ case. As an application, we give an explicit list of pairs (N, D) for which the Shimura curve $X_0^D(N)_{/\mathbb{Q}}$ may fail to have a sporadic point.