Closed, Bounded Sets
(also Convex)

- **Lecture Three: 25 Jan., 2005**

When are optimum values guaranteed?
First glimpses of how to compute optimizers? (First & Second Deriv. Tests, Convex Functions)

Readings:
- Reserve Books: Chong/Zak: Optimization, Chapter 5
  Peressini: Sections 1.2, 1.3 & 2.1

- **Basic Result for Existence of Optimal Values:**
  The Bolzano-Weierstrass Guarantee

Bolzano-Weierstrass (19th century): A continuous function on a closed and bounded domain D has global extrema on D.

- **Terminology: Open, Closed, Bounded, Continuity**

A set D is open if each point x in D has an open ball centered at x, B(x,r), contained in D.

A set D is closed if its complement (all those points not in D) is open.
Equivalently, D contains all points, x, for which there is a sequence, x[k], of points in D for which the limit $\|x - x[k]\|$ is zero (as k "goes to infinity").

**Examples:**

A set S of real numbers is open precisely when it is the union of a countable collection of disjoint open intervals, (a,b). There is no such simple characterization in higher dimensions.

The set $C = \{1/n: \text{n is a positive integer}\} \cup \{0\}$ is closed. Its complement is $(-\infty,0) \cup (1/(n+1),1/n) \cup (1,\infty)$ (where the second union is over all positive integers). This is a countable collection of disjoint open intervals.

The set $\{(x,y): 2x + 3y - 6 < 0\}$ is open and $\{(x,y): 2x + 3y - 6 \leq 0\}$ is closed.
**Basic Properties:** The collection of open sets is closed under the operations of taking arbitrary unions and finite intersections.

**Exercise:** Show that open sets are not closed under the operation of taking an intersection of a countable number of open sets.

**Boundedness:** A set $D$ is **bounded** if it is contained in some ball.

A function $F: D \rightarrow \mathbb{R}^n$ is continuous provided the inverse image of every open ball is relatively open in $D$ (i.e. the intersection of an open set and $D$). Equivalently, if the limit of $x[k]$ is $x$, then $F[x[k]]$ must have limit $F[x]$.

**Example:** $F: C \rightarrow \mathbb{R}$, given by $F[1/n]=n$, $F[0]=0$ is not continuous. Hint: The inverse image of (-1,1) is $\{0\}$ which is not open in $C$. (Any open interval containing 0 and intersected with $C$ would contain open points in $C$).

However, all polynomial function, common transcendental functions (Sine, Exp, Log) and functions built from these by adding, multiplying, dividing and composing are all continuous on their domain of definition.

- **Fundamental Investigation**

Find these guaranteed global extrema
Optimization theory is the study of such methods
Classical Optimization in Dimensions Two, Three, . . .

- Vanishing Gradient, Hessian Condition
- Taylor Series in Several Variables

Convex Sets and Functions

A set is **convex** provided the line segment from x to y, \{λ x + (1-λ)y: 0≤λ≤1\}, is in C whenever x and y are in C.

A real-valued function \( f: C \rightarrow \mathbb{R} \), is **convex** provided

\[
f[λ x + (1-λ)y] \leq λ f[x] + (1-λ) f[y],
\]

whenever 0≤λ≤1 and both x and y are in C.

- Basic Fact for Convex Functions

If the Hessian of f is positive semidefinite on C then f is convex.